An introduction to totally disconnected locally compact groups

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Introduction

For G a locally compact group, the connected component of the identity, denoted by G°, is a closed normal subgroup. We thereby obtain a short exact sequence of topological groups

$$\{1\} \to G^\circ \to G \to G/G^\circ \to \{1\}$$

where G/G° is the group of left cosets endowed with the quotient topology. The group G° is a connected locally compact group, and the group of components G/G° is a totally disconnected locally compact (t.d.l.c.) group. The study of locally compact groups therefore in principle, although not always in practice, reduces to studying connected locally compact groups and t.d.l.c. groups.

Many deep, general results have been discovered for connected locally compact groups over the last century. For instance, the connected locally compact groups are inverse limits of Lie groups, by the celebrated solution to Hilbert's fifth problem. The t.d.l.c. groups, on the other hand, long resisted a general theory. There were several early, promising results, the most compelling due to D. van Dantzig and H. Abels, but these results largely failed to ignite an active program of research. The indifference of the mathematical community seems to have arose from an inability to find a coherent metamathematical perspective via which to view the many disparate examples, which include both the profinite groups and the discrete groups. The insight, due to G. Willis [15], M. Burger and S. Mozes [4], and P.-E. Caprace and N. Monod [5], giving rise to a general theory is to study the *interactions between algebraic, geometric, and topological structure*. To put it another way, the modern theory of t.d.l.c. groups views the t.d.l.c. groups as simultaneously *geometric groups and topological groups* and investigates the connection between the geometric structure and topological structure. This perspective gives the profinite groups and the discrete groups a special status as basic building blocks, since the profinite groups are trivial as geometric groups and the discrete groups are trivial as topological groups.

This book covers what this author views as the central results in the theory of t.d.l.c. groups. We aim to present in full and clear detail the basic theorems and techniques a graduate student or researcher will need to study t.d.l.c. groups.

Prerequisites. The reader should have the mathematical maturity of a first or second year graduate student. We assume a working knowledge of abstract algebra, point-set topology, and functional analysis. The ideal reader will have taken graduate courses in abstract algebra, point-set topology, and functional analysis.

A word on second countability

In this text, we assume our groups are second countable whenever convenient. The theory of t.d.l.c. groups essentially reduces to studying second countable groups, so little generality is lost.

In the setting of locally compact groups, second countability admits a useful characterization.

Definition. A topological space is **Polish** if it is separable and admits a complete metric which induces the topology.

Fact. *The following are equivalent for a locally compact group G:*

(1) G is Polish.

(2) G is second countable.

(3) G is metrizable and K_σ - i.e. has a countable exhaustion by compact sets.

We may thus use the term "Polish" in place of "second countable" in the setting of locally compact groups. We will do so, because we will from time to time require results for Polish groups. (The class of Polish groups is a natural family of topological groups which is often studied in descriptive set theory and model theory.)

Notations

A **topological group** is a group endowed with a topology such that the group operations are continuous. All topological groups and spaces are taken to be Hausdorff. Groups are typically written multiplicatively. We use "t.d.", "l.c.", and "s.c." for "totally disconnected", "locally compact", and "second countable", respectively.

For H a closed subgroup of a topological group G, G/H denotes the space of left cosets, and $H\backslash G$ denotes the space of right cosets. We shall primarily consider left coset spaces. All quotient spaces of cosets are given the quotient topology. The center of G is denoted by $Z(G)$. For any subset $K \subseteq G$, $C_G(K)$ is the collection of elements of G that centralize every element of K. We denote the collection of elements of G that normalize K by $N_G(K)$. The topological closure of K in G is denoted by $\overline{K}$. For $A, B \subseteq G$, we put

$$\begin{aligned} A^B &:= \{bab^{-1} \mid a \in A \text{ and } b \in B\}, \\ [A, B] &:= \langle aba^{-1}b^{-1} \mid a \in A \text{ and } b \in B\rangle, \text{ and} \\ A^n &:= \{a_1 \dots a_n \mid a_i \in A\} \end{aligned}$$

For $k \geq 1$, $A^{\times k}$ denotes the k-th Cartesian power. For $a, b \in G$, $[a, b] := aba^{-1}b^{-1}$.

We denote a group G acting on a set X by $G \curvearrowright X$. Groups are always taken to act on the left. For a subset $F \subseteq X$, we denote the pointwise stabilizer of F in G by $G_{(F)}$. The setwise stabilizer is denoted by $G_{\{F\}}$.

We use the notation $\forall^\infty$ to stand for the phrase "for all but finitely many."

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Chapter 1

Topological Structure

1.1 van Dantzig's theorem

In a topological group, the topology is determined by a neighborhood basis at the identity. Having a collection $\{U_\alpha\}_{\alpha\in I}$ of arbitrarily small neighborhoods of the identity, we obtain a collection of arbitrarily small neighborhoods of any other group element g by forming $\{gU_\alpha\}_{\alpha\in I}$. We thus obtain a basis

$$\mathcal{B} := \{gU_\alpha \mid g \in G \text{ and } \alpha \in I\}$$

for the topology on G. We, somewhat abusively, call the collection $\{U_\alpha\}_{\alpha\in I}$ a **basis of identity neighborhoods** for G.

The topology of a totally disconnected locally compact (t.d.l.c.) group admits a well-behaved basis of identity neighborhoods; there is no need for the Polish assumption here. Isolating this basis requires a couple of classical results from point-set topology.

A topological space is **totally disconnected** if every connected subset has at most one element. A space is **zero dimensional** if it admits a basis of clopen sets; a **clopen** set is both closed and open. Zero dimensional spaces are totally disconnected, but in general the converse does not hold. For locally compact spaces, however, the converse does hold. This fact is well-known, but let us give a proof, as the techniques are informative. Recall that we take all topological spaces to be Hausdorff.

Lemma 1.1. *Let X be a compact space.*

1. *If C and D are non-empty closed subsets such that $C \cap D = \emptyset$, then*

*there are disjoint open sets U and V such that $C \subseteq U$ and $D \subseteq V$. That is, X is a **normal** topological space.*

2. *If $x \in X$ and A is the intersection of all clopen subsets of X containing x, then A is connected.*

Proof. For (1), let us first fix $c \in C$. Since X is Hausdorff, for each $d \in D$, there are disjoint open sets O_d and P_d such that $c \in O_d$ and $d \in P_d$. The set D is compact, so there is a finite collection $d_1, \dots, d_n$ of elements of D such that $D \subseteq \bigcup_{i=1}^n P_{d_i}$. We now see that $U_c := \bigcap_{i=1}^n O_{d_i}$ and $V_c := \bigcup_{i=1}^n P_{d_i}$ are disjoint open sets such that $c \in U_c$ and $D \subseteq V_c$.

For each $c \in C$, the set U_c is an open set that contains c, and V_c is an open set that is disjoint from U_c with $D \subseteq V_c$. As C is compact, there is a finite collection $c_1, \dots, c_m$ of elements of C such that $C \subseteq U := \bigcup_{i=1}^m U_{c_i}$. On the other hand, $D \subseteq V_{c_i}$ for each $1 \leq i \leq m$, so $D \subseteq V := \bigcap_{i=1}^m V_{c_i}$. The sets U and V satisfy the claim.

For (2), suppose that $A = C \cup D$ with C and D open in A and $C \cap D = \emptyset$; note that both C and D are closed in X. Applying part (1), we may find disjoint open sets U and V of X such that $C \subseteq U$ and $D \subseteq V$. Let $\{C_\alpha \mid \alpha \in I\}$ list the set of clopen sets of X that contain x. The intersection

$$\bigcap_{\alpha \in I} C_\alpha \cap (X \setminus (U \cup V))$$

is empty, so there is some finite collection $\alpha_1, \dots, \alpha_k$ in I such that $H := \bigcap_{i=1}^n C_{\alpha_i} \subseteq U \cup V$. We may thus write $H = H \cap U \cup H \cap V$, and since H is clopen, both $H \cap U$ and $H \cap V$ are clopen. The element x must be a member of one of $H \cap U$ or $H \cap V$; without loss of generality, we assume $x \in H \cap U$. The set A is the intersection of all clopen sets that contain x, so $A \subseteq H \cap U$. The set A is then disjoint from V which contains D, so D is empty. We conclude that A is connected. □

Lemma 1.2. *A totally disconnected locally compact space X is zero dimensional.*

Proof. Say that X is a totally disconnected locally compact space. Let $O \subseteq G$ be a compact neighborhood of $x \in G$ and say that $x \in U \subseteq O$ with U open in G. The set O is a totally disconnected compact space under the subspace topology. Letting $\{C_i\}_{i \in I}$ list clopen sets of O containing x, Lemma 1.1

ensures that $\bigcap_{i\in I} C_i = \{x\}$. The intersection $\bigcap_{i\in I} C_i \cap (O \setminus U)$ is empty, so there is $i_1, \dots, i_k$ such that $\bigcap_{j=1}^k C_{i_j} \subseteq U$. The set $\bigcap_{j=1}^k C_{i_j}$ is closed in O, so it is closed in G. On the other hand, $\bigcap_{j=1}^k C_{i_j}$ is open in the subspace topology on O, so there is $V \subseteq G$ open such that $V \cap O = V \cap U = \bigcap_{j=1}^k C_{i_j}$. We conclude that $\bigcap_{j=1}^k C_{i_j}$ is clopen in X. Hence, X is zero dimensional. □

That t.d.l.c. spaces are zero dimensional gives a canonical basis of identity neighborhoods for a t.d.l.c. group.

Theorem 1.3 (van Dantzig). *A t.d.l.c. group admits a basis at* 1 *of compact open subgroups.*

Proof. Let V be a neighborhood of 1 in G. By Lemma 1.2, G admits a basis of clopen sets at 1. We may thus find $U \subseteq V$ a compact open neighborhood of 1; we may take U to be symmetric since the inversion map is continuous.

For each $x \in U$, there is an open set W_x containing 1 with $xW_x \subseteq U$ and an open symmetric set L_x containing 1 with $L_x^2 \subseteq W_x$. The compactness of U ensures that $U \subseteq x_1L_{x_1} \cup \dots \cup x_kL_{x_k}$ for some $x_1, \dots, x_k$. Putting $L := \bigcap_{i=1}^k L_{x_i}$, we have

$$UL \subseteq \bigcup_{i=1}^k x_iL_{x_i}L \subseteq \bigcup_{i=1}^k x_iL_{x_i}^2 \subseteq \bigcup_{i=1}^k x_iW_{x_i} \subseteq U$$

We conclude that $UL \subseteq U$.

Induction on n shows that $L^n \subseteq U$ for all $n \geq 0$: if $L^n \subseteq U$, then $L^{n+1} = L^nL \subseteq UL \subseteq U$. The union $W := \bigcup_{n\geq 0} L^n$ is then contained in U. Since L is symmetric, W is an open subgroup of the compact open set U. As the compliment of W is open, W is indeed clopen, and therefore, W is a compact open subgroup of G contained in V. The theorem now follows. □

Any compact totally disconnected group is profinite; see the notes section of this Chapter. As a consequence of van Dantzig's theorem, we obtain the following.

Corollary 1.4. *A t.d.l.c. group admits a basis at* 1 *of open profinite subgroups.*

Notation 1.5. For a t.d.l.c. group G, we denote the collection of compact open subgroups by $\mathcal{U}(G)$.

We are primarily interested in t.d.l.c. Polish groups. For such groups G, the set $\mathcal{U}(G)$ is rather small.

Lemma 1.6. *If G is a t.d.l.c. Polish group, then $\mathcal{U}(G)$ is countable.*

Proof. Since G is second countable, we may fix a countable dense subset D of G. Applying van Dantzig's theorem, we may additionally fix a decreasing sequence $(U_i)_{i\in\mathbb{N}}$ of compact open subgroups giving a basis of identity neighborhoods.

For $V \in \mathcal{U}(G)$, there is i such that $U_i \leq V$. The subgroup V is compact, so U_i is of finite index in V. We may then find coset representatives $v_1, \dots, v_m$ such that $V = \bigcup_{j=1}^m v_j U_i$. For each v_j, there is $d_j \in D$ for which $d_j \in v_j U_i$, since the set D is dense in G. Therefore, $V = \bigcup_{j=1}^m d_j U_i$. We conclude that $\mathcal{U}(G)$ is contained in the collection of subgroups which are generated by $U_i \cup F$ for some $i \in \mathbb{N}$ and finite $F \subseteq D$. Hence, $\mathcal{U}(G)$ is countable. $\square$

A striking feature of the topological structure of t.d.l.c. Polish groups is that there are very few homeomorphism types of the underlying topological space.

A topological space is called **perfect** if it has no isolated points. A classical result of Brouwer shows that for a certain class of topological spaces there is exactly one perfect space up to homeomorphism.

Fact 1.7 (Brouwer)**.** *Any two non-empty compact Polish spaces which are perfect and zero dimensional are homeomorphic to each other.*

The Cantor space, denoted by $\mathcal{C}$, is thus the unique Polish space that is compact, perfect, and totally disconnected. Brouwer's theorem allows us to identify exactly the homeomorphism types of t.d.l.c. Polish groups.

Theorem 1.8. *For G a t.d.l.c. Polish group, one of the following hold:*

1. *G is homeomorphic to an at most countable discrete topological space.*
2. *G is homeomorphic to $\mathcal{C}$.*
3. *G is homeomorphic to $\mathcal{C} \times \mathbb{N}$ with the product topology.*

Proof. If the topology on G is discrete, then G is at most countable, since it is Polish, so (1) holds. Let us suppose that G is non-discrete. If the topology on G is compact, then G is perfect, compact, and totally disconnected; see

Exercise 1.2. In this case, (2) holds. Let us then suppose that G is neither discrete nor compact. Via Theorem 1.3, we obtain a compact open subgroup $U \leq G$. The group G is second countable, so we can fix coset representatives $(g_i)_{i \in \mathbb{N}}$ such that $G = \bigsqcup_{i \in \mathbb{N}} g_i U$. For each $i \in \mathbb{N}$, the coset $g_i U$ is a perfect, compact, and totally disconnected topological space. Fixing a homeomorphism $\phi_i : g_i U \to \mathcal{C}$ for each $i \in \mathbb{N}$, one verifies the map $\phi : G \to \mathcal{C} \times \mathbb{N}$ by $\phi(x) := (\phi_i(x), i)$ when $x \in g_i U$ is a homeomorphism. Hence, (3) holds. $\square$

Remark 1.9. Contrary to the setting of connected locally compact groups, Theorem 1.8 shows that there is no hope of using primarily the topology to investigate the structure of a t.d.l.c. Polish group. One must consider the algebraic structure and, as will be introduced later, the geometric structure in an essential way.

1.2 Isomorphism theorems

The usual isomorphism theorems for groups hold in the setting of l.c. groups with slight modification. We state these results for t.d.l.c. Polish groups, but they hold in somewhat more generality.

Our proofs require the classical Baire category theorem. Let X be a Polish space and $N \subseteq X$. We say N is **nowhere dense** if $\overline{N}$ has empty interior. We say $M \subseteq X$ is **meagre** if M is a countable union of nowhere dense sets.

Fact 1.10 (Baire Category Theorem)**.** *If X is a Polish space and $U \subseteq X$ is a non-empty open set, then U is non-meagre.*

Recall that an **epimorphism** from a group G to a group H is a surjective homomorphism.

Theorem 1.11 (First isomorphism theorem)**.** *Suppose that G and H are t.d.l.c. Polish groups with $\phi : G \to H$ a continuous epimorphism. Then ϕ is an open map. Further, the induced map $\tilde{\phi} : G/\ker(\phi) \to H$ given by $g\ker(\phi) \mapsto \phi(g)$ is an isomorphism of topological groups.*

Proof. Suppose that $B \subseteq G$ is open and fix $x \in B$. We may find $U \in \mathcal{U}(G)$ such that $xU \subseteq B$. If $\phi(U)$ is open, then $\phi(xU) = \phi(x)\phi(U) \subseteq \phi(B)$ is open. The map ϕ is thus open if $\phi(U)$ is open for every $U \in \mathcal{U}(G)$.

Fix $U \in \mathcal{U}(G)$. As G is second countable, we may find $(g_i)_{i\in\mathbb{N}}$ a countable set of left coset representatives for U in G. Hence,

$$H = \bigcup_{i\in\mathbb{N}} \phi(g_iU) = \bigcup_{i\in\mathbb{N}} \phi(g_i)\phi(U).$$

The subgroup U is compact, so $\phi(g_iU)$ is closed. The Baire category theorem then implies that $\phi(g_i)\phi(U)$ is non-meagre for some i. Multiplication by $\phi(g_i)$ is a homeomorphism of G, so $\phi(U)$ is non-meagre. The group $\phi(U)$ thus has a non-empty interior, and it follows that $\phi(U)$ is open in H.

For the second claim, it suffices to show $\tilde{\phi}$ is continuous since $\tilde{\phi}$ is bijective and our previous discussion insures it is an open map. Taking $O \subseteq H$ open, $\phi^{-1}(O)$ is open, since ϕ is continuous. Letting $\pi : G \to G/\ker(\phi)$ be the usual projection, the map π is a an open map. We conclude that

$$\pi(\phi^{-1}(O)) = \tilde{\phi}^{-1}(O)$$

is open in $G/\ker(\phi)$. Hence, $\tilde{\phi}$ is continuous. □

Corollary 1.12. *Suppose that G is a t.d.l.c. Polish group. If $\psi : G \to G$ is a continuous group isomorphism, then ψ is an automorphism of G as a topological group. That is, ψ^{-1} is continuous.*

Theorem 1.13 (Second isomorphism theorem)**.** *Suppose that G is a t.d.l.c. Polish group, $A \leq G$ is a closed subgroup, and $H \trianglelefteq G$ is a closed normal subgroup. If AH is closed, then $AH/A \simeq A/A \cap H$ as topological groups.*

Proof. Give AH and A the subspace topology and let $\iota : A \to AH$ be the obvious inclusion. The map ι is continuous. Letting $\pi : AH \to AH/H$ be the projection $x \mapsto xH$, the map π is a continuous epimorphism between t.d.l.c. Polish groups, so the composition $\pi \circ \iota : A \to AH/H$ is a continuous epimorphism. The first isomorphism theorem now implies $A/A\cap H \simeq AH/H$ as topological groups. □

In the second isomorphism theorem, AH must be closed to apply the first isomorphism theorem. If AH is not closed, then AH/H is not a locally compact group, so the first isomorphism theorem does not apply.

Theorem 1.14 (Third isomorphism theorem)**.** *Suppose that G and H are t.d.l.c. Polish groups with $\phi : G \to H$ a continuous epimorphism. If $N \trianglelefteq H$ is a closed normal subgroup, then $G/\phi^{-1}(N)$, H/N, and*

$$(G/\ker(\phi)) \,/\, \left(\phi^{-1}(N)/\ker(\phi)\right)$$

are all isomorphic as topological groups.

Proof. Let $\pi : H \to H/N$ be the usual projection; note that π is open and continuous. Applying the first isomorphism theorem to $\pi \circ \phi : G \to H/N$, we deduce that $G/\phi^{-1}(N) \simeq H/N$ as topological groups.

Applying the first isomorphism theorem to $\phi : G \to H$, the induced map $\tilde{\phi} : G/\ker(\phi) \to H$ is an isomorphism of topological groups. The composition $\pi \circ \tilde{\phi}$ is thus a continuous epimorphism with $\ker(\pi \circ \tilde{\phi}) = \phi^{-1}(N)/\ker(\phi)$. We conclude that

$$(G/\ker(\phi)) / \left(\phi^{-1}(N)/\ker(\phi)\right) \simeq H/N$$

as topological groups. □

Let us close this subsection with a useful characterization of continuous homomorphisms. Recall that a function $f : X \to Y$ between topological spaces is **continuous at** $x \in X$ if for every open neighborhood V of $f(x)$ there is an open neighborhood U of x such that $f(U) \subseteq V$. One can then define a function to be continuous if it is continuous at every point. In the setting of topological groups, one only needs to only check continuity at 1.

Proposition 1.15. *Suppose that G and H are topological groups with $\phi : G \to H$ a continuous homomorphism. Then ϕ is continuous if and only if ϕ is continuous at* 1*.*

Proof. The forward implication is immediate. Conversely, suppose ϕ is continuous at 1, fix $g \in G$, and let V be an open neighborhood of $\phi(g)$ in H. The translate $\phi(g^{-1})V$ is then an open neighborhood of 1, so we may find U an open set containing 1 such that $\phi(U) \leq \phi(g^{-1})V$. The set gU is an open neighborhood of g, and moreover,

$$\phi(gU) \leq \phi(g)\phi(g^{-1})V \leq V.$$

We conclude that ϕ is continuous at every $g \in G$, so ϕ is continuous. □

1.3 Locally finite graphs

Automorphism groups of locally finite connected graphs give a large and natural family of examples of t.d.l.c. groups. These examples furthermore give insight into the topological structure of t.d.l.c. groups, and in Chapter 3, we shall see that these examples are indeed integral to the theory of t.d.l.c. groups. We here carefully define this family and prove show they are indeed t.d.l.c. groups.

Definition 1.16. A **graph** Γ is a pair $(V\Gamma, E\Gamma)$ where $V\Gamma$ is a set and $E\Gamma$ is a collection of distinct pairs of elements from $V\Gamma$. We call $V\Gamma$ the set of **vertices** of Γ and $E\Gamma$ the set of **edges** of Γ.

Remark 1.17. One may alternatively consider, as logicians often do, $E\Gamma$ as a relation on $V\Gamma$. We discourage this perspective, because we shall later, in Chapter 4, need to modify our definition of a graph to allow for multiple edges and loops. This modification will be a natural extension of the definition given here.

For a vertex v, we define $E(v)$ to be the collection of edges e such that $v \in e$. A graph is **locally finite** if $|E(v)| < \infty$ for every $v \in V\Gamma$. A **path** p is a sequence of vertices $v_1, \dots, v_n$ such that $\{v_i, v_{i+1}\} \in E\Gamma$ for each $i < n$. The length of p, denoted by $l(p)$, is $n-1$. The length counts the number of edges used in the path. (Think about why we do not define the length to be the number of vertices in a path.) A least length path between two vertices is called a **geodesic**.

We say that a graph is **connected** if there is a path between any two vertices. Connected graphs are metric spaces under the graph metric: The **graph metric** on a connected graph Γ is

$$d_\Gamma(v,u) := \min\{l(p) \mid p \text{ is a path connecting } v \text{ to } u\}.$$

For $v \in V\Gamma$ and $k \geq 1$, the k**-ball** around v is defined to be $B_k(v) := \{w \in V\Gamma \mid d_\Gamma(v,w) \leq k\}$ and the k**-sphere** is defined to be $S_k(v) := \{w \in V\Gamma \mid d_\Gamma(v,w) = k\}$. When we wish to emphasize the graph in which we are taking $B_k(v)$ and $S_k(v)$, we write $B_k^\Gamma(v)$ and $S_k^\Gamma(v)$.

For graphs Γ and Δ, a **graph isomorphism** is a bijection $\psi : V\Gamma \to V\Delta$ such that $\{g(v), g(w)\} \in E\Delta$ if and only if $\{v,w\} \in E\Gamma$. An **automorphism** of a graph Γ is an isomorphism $\psi : \Gamma \to \Gamma$. The collection of automorphisms forms a group, and it is denoted by $\mathrm{Aut}(\Gamma)$. When Γ is a connected graph, the automorphism group is the same as the isometry group of Γ, when Γ is regarded as a metric space under the graph metric.

The group $\mathrm{Aut}(\Gamma)$ admits a natural topology, which we shall see makes it into a topological group. For finite tuples $\overline{a} := (a_1, \ldots, a_n)$ and $\overline{b} := (b_1, \ldots, b_n)$ of vertices, define

$$\Sigma_{\overline{a},\overline{b}} := \{g \in \mathrm{Aut}(\Gamma) \mid g(a_i) = b_i \text{ for } 1 \leq i \leq n\}.$$

The collection $\mathcal{B}$ of sets $\Sigma_{\overline{a},\overline{b}}$ as $\overline{a}$ and $\overline{b}$ run over finite sequences of vertices forms a basis for a topology on $\mathrm{Aut}(\Gamma)$. The topology generated by $\mathcal{B}$ is called the **pointwise convergence topology** (exercise 1.23 motivates this terminology). The pointwise convergence topology is sometimes called the **permutation topology**.

Proposition 1.18. *Let Γ be a graph. Equipped with the pointwise convergence topology,* $\mathrm{Aut}(\Gamma)$ *is a topological group.*

Proof. We must show that composition and inversion are continuous under the pointwise convergence topology. For inversion, take a basic open set $\Sigma_{\overline{a},\overline{b}}$. The preimage of $\Sigma_{\overline{a},\overline{b}}$ under the inversion map is $\Sigma_{\overline{b},\overline{a}}$, hence inversion is continuous.

Let $m : G \times G \to G$ be the multiplication map. Fix a basic open set $\Sigma_{\overline{a},\overline{b}}$ with $\overline{a} = (a_1, \ldots, a_n)$ and $\overline{b} = (b_1, \ldots, b_n)$. Fix $(h, g) \in m^{-1}(\Sigma_{\overline{a},\overline{b}})$. We may find a tuple $\overline{c} = (c_1, \ldots, c_n)$ such that $g(a_i) = c_i$, and since $hg \in \Sigma_{\overline{a},\overline{b}}$, it must be the case that $h(c_i) = b_i$. The open set $\Sigma_{\overline{c},\overline{b}} \times \Sigma_{\overline{a},\overline{c}}$ is then an open set containing (h, g), and it is contained in $m^{-1}(\Sigma_{\overline{a},\overline{b}})$. Hence, m is continuous. $\square$

Remark 1.19. We shall always assume the automorphism group of a graph is equipped with pointwise convergence topology.

From the definition of the pointwise convergence topology, we immediately deduce the following.

Proposition 1.20. *Let Γ be a graph. If Γ is countable, then the pointwise convergence topology on* $\mathrm{Aut}(\Gamma)$ *is second countable.*

For Γ a graph, set $G := \mathrm{Aut}(\Gamma)$ and for $F \subseteq V\Gamma$ finite, define $G_{(F)}$ to be the pointwise stabilizer of the set F in G. The set $G_{(F)}$ is a basic open set, and

$$\mathcal{F} := \{G_{(F)} \mid F \subseteq V\Gamma \text{ with } |F| < \infty\}$$

is a basis at the identity. The sets $G_{(F)}$ are subgroups, so $\mathcal{F}$ in fact is a basis of clopen subgroups. Since a basis for the topology on $\mathrm{Aut}(\Gamma)$ is given by cosets of the elements of $\mathcal{F}$, we have proved the following proposition.

Proposition 1.21. *Let Γ be a graph. The pointwise convergence topology on* $\mathrm{Aut}(\Gamma)$ *is zero dimensional. In particular, the pointwise convergence topology is totally disconnected.*

To isolate the desired family of t.d.l.c. Polish groups, we require a notion of a Cauchy sequence in a topological group. While topological groups do not have a metric in general, there is nonetheless a notion of "close together" since we can consider the "difference" of two group elements.

Definition 1.22. Let G be a topological group and $\mathcal{B}$ a basis of identity neighborhoods. A sequence $(g_i)_{i\in\mathbb{N}}$ of elements of G is a **Cauchy sequence** if for every $B\in\mathcal{B}$ there is $N\in\mathbb{N}$ such that $g_i^{-1}g_j\in B$ and $g_jg_i^{-1}\in B$ for all $i,j\geq N$.

The reader should work Exercise 1.7, which verifies that the definition of a Cauchy sequence does not depend on the choice of $\mathcal{B}$.

Lemma 1.23. *Let Γ be a graph. If $(g_i)_{i\in\mathbb{N}}$ is a Cauchy sequence in* $\mathrm{Aut}(\Gamma)$*, then there is $g\in\mathrm{Aut}(\Gamma)$ such that $(g_i)_{i\in\mathbb{N}}$ converges to g.*

Proof. For $v\in V\Gamma$, fix $N_v\geq 1$ such that $g_i^{-1}g_j(v)=v$ and $g_ig_j^{-1}(v)=v$ for all $i,j\geq N_v$. We may find such an N_v since $(g_i)_{i\in\mathbb{N}}$ is a Cauchy sequence. Define a function $g:V\Gamma\to V\Gamma$ by $g(v):=g_{N_v}(v)$.

We first argue that g is a permutation of $V\Gamma$. Suppose that $g(v)=g(w)$. Thus, $g_{N_v}(v)=g_{N_w}(w)$. Taking $M:=\max\{N_v,N_w\}$, we see that $g_M(v)=g_{N_v}(v)$ and $g_M(w)=g_{N_w}(w)$. It is then the case that $g_M(v)=g_M(w)$, and since g_M is a bijection, $v=w$. Hence, g is injective.

To see that g is surjective, take $w\in V\Gamma$. Observe that $g_ig_j^{-1}(w)=w$ for all $i,j\geq N_w$ and set $v:=g_j^{-1}(w)$ for some fixed $j>N_w$. Taking M greater than both N_v and N_w, we have that $g_M(v)=g_{N_v}(v)=g(v)$. On the other hand, that $M\geq N_w$ ensures that $g_M(v)=g_Mg_j^{-1}(w)=w$. We conclude that $g(v)=w$, and thus, g is bijective.

We finally argue that g respects the graph structure. Fix $\{v,w\}$ a distinct pair of vertices. Taking M greater than both N_v and N_w. We see that $g(v)=g_{N_v}(v)=g_M(v)$ and $g(w)=g_{N_w}(w)=g_M(w)$. Since g_M preserves the graph structure, we conclude that $\{v,w\}\in E\Gamma$ if and only if $\{g(v),g(w)\}\in E\Gamma$. Hence, $g\in\mathrm{Aut}(\Gamma)$.

We leave that $g_i\to g$ as an exercise. □

Theorem 1.24. *Let Γ be a graph. If Γ is locally finite and connected, then* $\mathrm{Aut}(\Gamma)$ *is a t.d.l.c. Polish group.*

Proof. Set $G := \mathrm{Aut}(\Gamma)$, fix a vertex $v \in V\Gamma$, and take the vertex stabilizer $G_{(v)}$. For each $k \geq 1$, set $S_k := S_k(v)$, where $S_k(v)$ is the k-sphere around v. Since Γ is locally finite, it follows by induction on k that S_k is finite for every $k \geq 1$.

The group $G_{(v)}$ acts on each S_k as a permutation, so we obtain a family of homomorphisms $\phi_k : G \to \mathrm{Sym}(S_k)$. Define $\Phi : G_{(v)} \to \prod_{k\geq 1} \mathrm{Sym}(S_k)$ by $\Phi(g) := (\phi_k(g))_{k\geq 1}$. The map Φ is a homomorphism, since each ϕ_k is a homomorphism. As Γ is connected, $V\Gamma = \{v\} \cup \bigcup_{k\geq 1} S_k$, so Φ is also injective.

We next argue that Φ is continuous. In view of Proposition 1.15, it suffices to check that Φ is continuous at 1. Setting $L := \prod_{k\geq 1} \mathrm{Sym}(S_k)$, a basis at 1 for L is given by the subgroups

$$\Delta_n := \{(r_i)_{i\geq 1} \in L \mid r_i = 1 \text{ for all } i \leq n\}.$$

Fix $n \geq 1$. The pointwise stabilizer $G_{(\bigcup_{i=1}^n S_i)}$ is an open subgroup of $G_{(v)}$, and $\Phi(U) \leq \Delta_n$. The map Φ is thus continuous at 1, so Φ is continuous.

Take $A \subseteq G_{(v)}$ a closed set and suppose that $(\Phi(a_i))_{i\in\mathbb{N}}$ with $a_i \in A$ is a convergent sequence in L. The sequence $(\Phi(a_i))_{i\in\mathbb{N}}$ is a Cauchy sequence in L, so for every Δ_n, there is N such that $\Phi(a_i^{-1}a_j) \in \Delta_n$ and $\Phi(a_j a_i^{-1}) \in \Delta_n$ for any $i, j \geq N$. The elements $a_j a_i^{-1}$ and $a_i^{-1}a_j$ therefore fix $B_n(v)$ pointwise for all $i, j \geq N$. As the collection of pointwise stabilizers $\{G_{(B_n(v))} \mid n \geq 1\}$ form a basis at 1, we deduce that $(a_i)_{i\in\mathbb{N}}$ is a Cauchy sequence in $G_{(v)}$. Lemma 1.23 now supplies $a \in G$ such that $a_i \to a$. As A is closed, we indeed have that $a \in A$. Since Φ is continuous, we infer that $\Phi(a) = \lim_i \Phi(a_i)$, so $\Phi(A)$ is closed. The homomorphism Φ is thus a closed map and so a topological group isomorphism.

The pointwise stabilizer $G_{(v)}$ is isomorphic to a closed subgroup of L, so $G_{(v)}$ is compact. The group G is thus locally compact. Propositions 1.21 and 1.20 now ensure that G is a t.d.l.c. Polish group. □

It is a notoriously difficult problem to determine if the automorphism group of a locally finite connected graph is non-discrete. However, there are many cases where the geometry of the graph allows us to argue that the automorphism group is non-discrete. A large source of such examples are given by locally finite trees. A **tree** is a connected graph such that there are no cycles. A **cycle** is a path $p_1, \dots, p_n$ with $n > 1$ such that $p_i = p_j$ if and only if $\{i, j\} = \{1, n\}$. It is easy to produce many examples of locally finite trees with non-discrete automorphism groups; see Exercise 1.24.

1.4 The wreath product

Suppose G and H are groups and G acts on H by automorphisms. For a group acting on a second group, we denote the action of $g \in G$ on $h \in H$ by $g.h$; for a group action $G \curvearrowright X$ with X a set or topological space, we denote the action by $g(x)$. By classical results in abstract group theory, the Cartesian product $H \times G$ becomes a group under the multiplication

$$(h_1, g_1) \circ (h_2, g_2) := (h_1 \circ (g_1.h_2), g_1 \circ g_2)$$

and inversion

$$(h, g)^{-1} := (g^{-1}.h^{-1}, g^{-1}).$$

We wish to make $H \rtimes G$ into a topological group when both G and H are topological groups. To do so, we need a notion of continuity for a group action.

Definition 1.25. Suppose G is a topological group with an action on a topological space X. We say that the action is **continuous** if the action map $\alpha : G \times X \to X$ defined by $(g, x) \mapsto g(x)$ is continuous.

Continuous actions allow us to topologize a semi-direct product of topological groups.

Definition 1.26. Suppose that G and H are topological groups and G acts continuously on H by topological group automorphisms. The **semi-direct product** is the usual semi-direct product group $H \rtimes G$ arising from the action of G on H that is further equipped with the product topology.

Our proof of the next proposition implicitly uses the following basic fact from point-set topology: A function $f : Z \to X \times Y$ is continuous if and only if the composition $\pi_i \circ f$ is continuous for each $i \in \{0, 1\}$, where π_i is the projection onto the i-th coordinate.

Proposition 1.27. *Suppose that G and H are topological groups and G acts continuously on H by topological group automorphisms. The semi-direct product $H \rtimes G$ is then a topological group. If additionally both G and H are Polish groups or t.d.l.c. groups, then $H \rtimes G$ is also a Polish group or a t.d.l.c. group, respectively.*

Proof. We argue that multiplication is continuous; that inversion is continuous follows similarly. Say that $m : (H \rtimes G)^2 \to H \rtimes G$ is the multiplication map. Recalling that $(H \rtimes G)^2 = H \times G \times H \times G$, we see that $m((h_1, g_1, h_2, g_2)) = (h_1 \circ (g_1.h_2), g_1 \circ g_2)$. We now decompose m into continuous maps. Let $m_1 : H \times G \times H \times G \to H \times G \times H \times G$ by

$$m_1((h_1, g_1, h_2, g_2)) := (h_1, g_1, g_1.h_2, g_2).$$

Since the action is continuous, the function m_1 is continuous. Let $m_2 : H \times G \times H \times G \to H \times G \times H \times G$ by

$$m_2((h_1, g_1, h_2, g_2)) := (h_1, g_1, h_2, g_1 \circ g_2).$$

Since multiplication is continuous, m_2 is again a continuous map. The last continuous function needed is $m_3 : H \times G \times H \times G \to H \times G$ by

$$m_3((h_1, g_1, h_2, g_2)) := (h_1 \circ h_2, g_2).$$

We now see that $m_3 \circ m_2 \circ m_1 = m$, so m is continuous, as desired.

The additional claims are immediate since the classes of Polish spaces and t.d.l.c. spaces are closed under finite direct products. $\square$

Building a wreath product requires a notion of infinite direct product or sum. What we shall in fact obtain is a notion of a product that sits between the direct sum and the direct product.

Notation 1.28. We use the notation $\forall^\infty$ to stand for the phrase "for all but finitely many."

Definition 1.29. Let $(G_x)_{x \in X}$ be a family of t.d.l.c. groups and $(U_x)_{x \in X}$ a sequence such that $U_x \in \mathcal{U}(G_x)$ for each $x \in X$. The **restricted direct product** of $(G_x)_{x \in X}$ over $(U_x)_{x \in X}$ is defined to be

$$\left\{ f : X \to \bigcup_{x \in X} G_x \mid \forall x \in X \; f(x) \in G_x \text{ and } \forall^\infty x \in X \; f(x) \in U_x \right\}.$$

and denoted by $\bigoplus_{x \in X} (G_x, U_x)$.

The set $\bigoplus_{x \in X}(G_x, U_x)$ is a group under pointwise multiplication. The group $\bigoplus_{x \in X} (G_x, U_x)$ is further admits a topology generated by the basis $\mathcal{B}$ consisting all sets of the form $\prod_{x \in X} O_x$ such that O_x is open in G_x for all x and $O_x = U_x$ for all but finitely many x. We call the topology generated by $\mathcal{B}$ the **restricted product topology**. We shall always consider $\bigoplus_{x \in X} (G_x, U_x)$ to be equipped with the restricted product topology.

Lemma 1.30. *Let $(G_x)_{x\in X}$ be a family of t.d.l.c. groups and $(U_x)_{x\in X}$ a sequence such that $U_x \in \mathcal{U}(G_x)$ for each $x \in X$. Equipping $\bigoplus_{x\in X}(G_x, U_x)$ with the restricted product topology, the following hold:*

(1) $\bigoplus_{x\in X}(G_x, U_x)$ is a t.d.l.c. group.

(2) $\prod_{x\in X} U_x$ is a compact open subgroup of $\bigoplus_{x\in X}(G_x, U_x)$.

(3) If X is countable and G_x is a t.d.l.c. Polish group for each $x \in X$, then $\bigoplus_{x\in X}(G_x, U_x)$ is a t.d.l.c. Polish group.

Proof. See Exercise 1.19. □

The restricted direct product depends on the choice of the U_x. For instance, let $(F_i)_{i\in\mathbb{Z}}$ list copies of a non-trivial finite group F. We now see

$$\bigoplus_{i\in\mathbb{Z}}(F_i, \{1\}) = \bigoplus_{i\in\mathbb{Z}} F_i \text{ and } \bigoplus_{i\in\mathbb{Z}}(F_i, F_i) = \prod_{i\in\mathbb{Z}} F_i.$$

This in particular shows the dependence of the product on the choice of the compact open subgroups: we obtain a discrete group on one hand and a compact group on the other! We can in fact do even better. Put

$$U_i := \begin{cases} F_i, & \text{if } i \leq 0 \\ \{1\}, & \text{otherwise.} \end{cases}$$

It is easy to verify $\bigoplus_{i\in\mathbb{Z}}(F_i, U_i)$ is non-compact and non-discrete but locally compact.

Proposition 1.31. *Suppose that G is a t.d.l.c. group, $U \in \mathcal{U}(G)$, and A is a discrete topological space. Let $(G_a)_{a\in A}$ and $(U_a)_{a\in A}$ list copies of G and U, respectively. If H is a t.d.l.c. group with a continuous action on A, then the action of H on $\bigoplus_{a\in A}(G_a, U_a)$ defined by $h.f(a) := f(h^{-1}.a)$ is a continuous action by topological group automorphisms*

Proof. Let $\alpha : H \times \bigoplus_{a\in A}(G_a, U_a) \to \bigoplus_{a\in A}(G_a, U_a)$ be the action map. It is easy to see the action is by group automorphisms. Fix $W \subseteq \bigoplus_{a\in A}(G_a, U_a)$ a basic open set. There is then a finite set $A_0 \subseteq A$ such that

$$W = \prod_{a\in A_0} Y_a \times \prod_{a\in A\setminus A_0} U_a.$$

By possibly reducing the size of W, we may assume that $Y_a = g_a W_a$ where $g_a \in G_a$ and W_a is an open subgroup of U_a.

Fix $(h, f) \in \alpha^{-1}(W)$. Set $V := \bigcap_{a \in A_0} W_a$, $B := h^{-1}.A_0$, and $L := V^B \times \prod_{a \in A \setminus B} U_a$. Set $O := H_{(A_0)}$ and note that O is an open subgroup of H because A_0 is finite and H acts on A continuously. We now consider the open set $Oh \times fL$. Take $(oh, fl) \in Oh \times fl$. For $a \in A_0$,

$$(oh).(fl)(a) = f(h^{-1}o^{-1}.a)l(h^{-1}o^{-1}.a) = f(h^{-1}.a)l(h^{-1}.a).$$

The element $f(h^{-1}.a)$ is in $g_a W_a$ and $l(h^{-1}.a)$ is an element of $V \leq W_a$. We conclude that $f(h^{-1}.a)l(h^{-1}.a) \in g_a W_a$.

For $a \notin A_0$, $f(h^{-1}o^{-1}.a) \in U_a$, since $o^{-1}.a \notin A_0$. On the other hand, $l(h^{-1}.a)$ is always in U_a, so we conclude that $(oh).(fl)(a) \in U_a$. Therefore, $(oh).(fl) \in W$, so α is continuous. $\square$

Definition 1.32. The action given in Proposition 1.31 is called the **shift action** of H on $\bigoplus_{a \in A}(G_a, U_a)$.

The shift action allows us to form the semi-direct product $\bigoplus_{a \in A}(G_a, U_a) \rtimes H$, thereby recovering a notion of wreath product.

Definition 1.33. Suppose that G and H are t.d.l.c. groups, U is a compact open subgroup of G, and A is a discrete topological space on which H acts continuously. Letting $(G_a)_{a \in A}$ and $(U_a)_{a \in A}$ list copies of G and U, respectively, the **wreath product** of G and H with respect to U and A is defined to be

$$G \wr_U (H, A) := \bigoplus_{a \in A} (G_a, U_a) \rtimes H$$

where $H \curvearrowright \bigoplus_{a \in A} (G_a, U_a)$ by shift.

Proposition 1.34. *Suppose that G and H are t.d.l.c. groups, U is a compact open subgroup of G, and A is a discrete topological space on which H acts continuously. Then the wreath product $G \wr_U (H, A)$ enjoys the following properties:*

1. *$G \wr_U (H, A)$ is a t.d.l.c. group.*

2. *If in addition G and H are Polish and A is countable, then $G \wr_U (H, A)$ is a t.d.l.c. Polish group.*

Proof. By definition, $G \wr_U (H, A) := \bigoplus_{a \in A} (G_a, U_a) \rtimes H$. The group H is assumed to be a t.d.l.c. group, and Lemma 1.30 ensures that $\bigoplus_{a \in A} (G_a, U_a)$ is a t.d.l.c. groups. Appealing to Proposition 1.31, the action of H on $\bigoplus_{a \in A}(G_a, U_a)$ is furthermore continuous, so by Proposition 1.27, we conclude that $G \wr_U (H, A)$ is a t.d.l.c. group, verifying (1).

For (2), we see by Lemma 1.30 that $\bigoplus_{a \in A}(G_a, U_a)$ is a t.d.l.c. Polish group. Proposition 1.27 then implies that $G \wr_U (H, A)$ is also a t.d.l.c. Polish group. □

In the setting of topological groups, there is a natural analogue of finite generation. The analogous statements for finitely generated groups and wreath products hold for compactly generated t.d.l.c. groups and wreath products.

Definition 1.35. A topological group G is **compactly generated** if there is a compact set K such that $\langle K \rangle = G$.

Proposition 1.36. *Suppose G and H are t.d.l.c. groups, U is a compact open subgroup of G, and A is a discrete topological space on which H acts continuously. If G and H are compactly generated and H has finitely many orbits on A, then $G \wr_U (H, A)$ is compactly generated.*

Proof. - to add □

Notes

Theorem 1.8 is particularly striking if one is accustomed to connected Lie groups. Information about to topology of a connected Lie group often gives deep insight into the structure of the group. For instance, two compact connected simple Lie groups are isomorphic as Lie groups if and only if they have the same homotopy type (see [2, Theorem 9.3]).

For an excellent discussion and proof of the Baire category theorem as well as a proof of Brouwer's theorem, the reader is directed to [10]. For the statements of the isomorphism theorems in full generality, we refer the reader to [9].

1.5 Exercises

Topological groups

Exercise 1.1. Let G be a group with normal subgroups L and K. Show if $L \cap K = \{1\}$, then L and K centralize each other.

Exercise 1.2. Let G be a non-discrete topological group. Show G is perfect as a topological space.

Exercise 1.3. Suppose that G is a topological group and V is an open neighborhood of g. Show there is an open set W containing 1 such that $WgW \subseteq V$.

Exercise 1.4. Let G be a topological group and $H \leq G$ a subgroup with non-empty interior. Show H is open and closed in G.

Exercise 1.5. Suppose that G is topological group and $H \trianglelefteq G$ is not closed. Show $\overline{H}$ is normal subgroup of G.

Exercise 1.6. Suppose that G and H are topological groups and $\phi : G \to H$ is a homomorphism. Show that ϕ is continuous if and only if ϕ is continuous at some $g \in G$.

Exercise 1.7. Show the definition of a Cauchy sequence does not depend on the choice of a basis of identity neighborhoods.

Exercise 1.8. Let G be a topological group and suppose $(g_i)_{i\in\mathbb{N}}$ is a convergent sequence. Show $(g_i)_{i\in\mathbb{N}}$ is a Cauchy sequence.

Locally compact groups

Exercise 1.9. Let G be a non-compact and non-discrete t.d.l.c. Polish group. Fix compact open subgroup $U \leq G$, coset representatives $(g_i)_{i\in\mathbb{N}}$ such that $G = \bigsqcup_{i\in\mathbb{N}} g_iU$, and a homeomorphism $\phi_i : g_iU \to \mathcal{C}$ for each $i \in \mathbb{N}$, where $\mathcal{C}$ is the Cantor set. Verify the map $\phi : G \to \mathcal{C} \times \mathbb{N}$ by $\phi(x) := (\phi_i(x), i)$ when $x \in g_iU$ is a homeomorphism.

Exercise 1.10. Let G be a locally compact group. Show the connected component of the identity is a closed normal subgroup of G.

Exercise 1.11. Let G be a compact t.d.l.c. group. Show G admits a basis at 1 of compact open *normal* subgroups and each of these subgroups is of finite index in G.

Exercise 1.12. Suppose that G is a t.d.l.c. group with $K \leq G$ a compact subgroup. Show there is a compact open subgroup U of G containing K.

Exercise 1.13. Suppose that G is a t.d.l.c. group and $K \trianglelefteq G$ is such that K and G/K are compact. Show G is compact. This shows the class of profinite groups is closed under group extension.

Exercise 1.14. Suppose that G is a t.d.l.c. Polish group, $K \subseteq G$ is compact, and $C \subseteq G$ is closed. Show KC is a closed subset of G.

Exercise 1.15. Suppose G is a t.d.l.c. Polish group and $U \in \mathcal{U}(G)$. Show if $K \subseteq G$, then $\overline{K \cap U} = \overline{K} \cap U$.

Exercise 1.16. Suppose G is a t.d.l.c. Polish group, $H \leq G$, and $U \in \mathcal{U}(G)$. Show if $H \cap U$ is closed in G, then H is closed in G.

Exercise 1.17. Suppose $(G_i)_{i\in\mathbb{N}}$ is a countable increasing union of t.d.l.c. Polish groups such that $G_i \leq_o G_{i+1}$ for each i. Define a collection τ of subsets of $\bigcup_{i\in\mathbb{N}} G_i$ by $A \in \tau$ if and only if $A \cap G_i$ is open in G_i for all $i \in \mathbb{N}$. Show τ is a topology on $\bigcup_{i\in I} G_i$ and verify $\bigcup_{i\in\mathbb{N}} G_i$ is a t.d.l.c. Polish group under this topology. We call τ the **inductive limit topology**.

Exercise 1.18. Let $(G_a)_{a\in A}$ be a sequence of t.d.l.c. groups and let $U_a \leq G_a$ be a compact open subgroup. Show that collection $\mathcal{B}$ of subsets of $\bigoplus_{a\in A} (G_a, U_a)$ of the form $\prod_{a\in A} O_a$ such that O_a is open in G_a for all a and $O_a = U_a$ for all but finitely many a is a basis for a topology on $\bigoplus_{a\in A}(G_a, U_a)$.

Exercise 1.19. Prove Lemma 1.30.

Exercise 1.20. Suppose that G is a t.d.l.c. group with a continuous action on a countable set X. Show $A \subseteq G$ is relatively compact - i.e., $\overline{A}$ is compact - if and only if for all $x \in X$ the set $A.x := \{a.x \mid a \in A\}$ is finite.

Graphs

Exercise 1.21. Let Γ be a graph and for any finite tuples $\overline{a} := (a_1, \dots, a_n)$ and $\overline{b} := (b_1, \dots, b_n)$ of vertices, define

$$\Sigma_{\overline{a},\overline{b}} := \{g \in \mathrm{Aut}(\Gamma) \mid g(a_i) = b_i \text{ for } 1 \leq i \leq n\}.$$

Show the collection $\mathcal{B}$ of the sets $\Sigma_{\overline{a},\overline{b}}$ as $\overline{a}$ and $\overline{b}$ range over finite tuples of vertices is a basis for a topology on $\mathrm{Aut}(\Gamma)$.

Exercise 1.22. Let Γ be a locally finite connected graph and fix $v \in V\Gamma$. For $k \geq 1$, show $B_k(v) := \{w \in V\Gamma \mid d_\Gamma(v, w) \leq k\}$ is finite.

Exercise 1.23. Let Γ be a graph and and $(g_i)_{i\in\mathbb{N}}$ be a sequence of elements from $\mathrm{Aut}(\Gamma)$. Show $(g_i)_{i\in\mathbb{N}}$ converges to some $g \in \mathrm{Aut}(\Gamma)$ if and only if for every finite set $F \subset V\Gamma$ there is N such that $g_i(x) = g(x)$ for all $x \in F$ and all $i \geq N$.

Exercise 1.24. For $n \geq 3$, let T_n be the tree such that $\deg(v) = n$ for every $v \in VT$; we call T_n the ***n*-regular tree**. Show $\mathrm{Aut}(T_n)$ is non-discrete.

Chapter 2

Haar Measure

The primary goal of this chapter is to establish the existence and uniqueness of a canonical measure on a t.d.l.c. Polish group, called the Haar measure. The Haar measure is a powerful tool, which allows the techniques from functional analysis to be applied to the study of t.d.l.c. Polish groups.

2.1 Functional analysis

Let us begin by recalling the basics of functional analysis; we shall restrict our discussion to Polish spaces. For X a Polish space, the collection of continuous functions $f : X \to \mathbb{C}$ is denoted by $C(X)$. The set $C(X)$ is a vector space under pointwise addition and scalar multiplication. The **support** of $f \in C(X)$ is $\mathrm{supp}(f) := \overline{\{x \in X \mid f(x) \neq 0\}}$. A function $f \in C(X)$ is **compactly supported** if $\mathrm{supp}(f)$ is compact. We denote the set of compactly supported functions in $C(X)$ by $C_c(X)$. A function $f \in C(X)$ is said to be **positive** if $f(X) \subseteq \mathbb{R}_{\geq 0}$. The collection of all positive functions in $C_c(X)$ is denoted by $C_c^+(X)$.

We can equip $C_c(X)$ with the following norm:

$$\|f\| := \max\{|f(x)| \mid x \in X\}.$$

This norm is called the **uniform norm** on $C_c(X)$. It induces a metric d on $C_c(X)$ defined by $d(f, g) := \|f - g\|$. The metric topology on $C_c(X)$ turns $C_c(X)$ into a topological vector space. That is to say, the vector space operations are continuous in this topology; see Exercise 2.5. We call this topology the **uniform topology**, and we shall always take $C_c(X)$ to be equipped with this topology.

When considering a Polish group G, the group G has a left and a right action on $C_c(G)$. The left action is given by $L_g(f)(x) := f(g^{-1}x)$, and the right action is given by $R_g(f)(x) := f(xg)$. One verifies if $f_i \in C_c(G)$ converges to some $f \in C_c(G)$, then $L_g(f_i) \to L_g(f)$ and $R_g(f_i) \to R_g(f)$ for any $g \in G$; see Exercise 2.6.

A **linear functional** on $C_c(X)$ is a linear function $\Phi : C_c(X) \to \mathbb{C}$. A functional is **positive** if the positive functions are taken to non-negative real numbers. Positive linear functionals are always continuous; see Exercise 2.7. If X is additionally compact, a functional Φ is called **normalized** if $\Phi(1_X) = 1$, where 1_X is the indicator function for X.

For a Polish group G, a linear functional Φ is **left-invariant** if $\Phi(L_g(f)) = \Phi(f)$ for all $f \in C_c(G)$ and $g \in G$, and it is **right-invariant** if $\Phi(R_g(f)) = \Phi(f)$ for all $f \in C_c(G)$ and $g \in G$.

For X a Polish space, a **sigma algebra** is a collection $\mathcal{S}$ of subsets of X such that $\mathcal{S}$ contains the empty set and is closed under taking complements, countable unions, and countable intersections. The **Borel sigma algebra** of X is the smallest sigma algebra that contains the open sets of X. The Borel sigma algebra is denoted by $\mathcal{B}(X)$ or just $\mathcal{B}$ when the space X is clear from context.

A measure μ defined on a Polish space X is called a **Borel measure** if μ is defined on a sigma algebra $\mathcal{C}$ that contains the Borel sigma algebra of X. In practice, the sigma algebra $\mathcal{C}$ is typically the sigma algebra generated by $\mathcal{B}$ along with all null sets; a **null set** is a subset of a Borel set with measure zero. A Borel measure is said to be **locally finite** if for every $x \in X$ there is a neighborhood U of x with finite measure. A **probability measure** is a measure such that the measure of the entire space is 1.

The measures we construct on t.d.l.c. Polish groups shall have two important properties.

Definition 2.1. A locally finite Borel measure μ on a Polish space X is called an **outer Radon measure** if the following hold:

(i) (Outer regularity) For all $A \in \mathcal{B}(X)$,
$$\mu(A) = \inf\{\mu(U) \mid U \supseteq A \text{ is open}\}.$$

(ii) (Inner regularity) For all $A \in \mathcal{B}(X)$ such that $\mu(A) < \infty$,
$$\mu(A) = \sup\{\mu(K) \mid K \subseteq A \text{ is compact}\}.$$

Outer Radon measures have good approximation properties. The measure of a set can be approximated from above by open sets and from below by compact sets.

Definition 2.2. A Borel measure μ on a Polish group G is **left-invariant** if $\mu(gA) = \mu(A)$ for all Borel sets $A \subseteq G$ and $g \in G$. We say the measure is **right-invariant** if $\mu(Ag) = \mu(A)$ for all Borel sets $A \subseteq G$ and $g \in G$.

For a locally compact Polish space X, linear functionals on $C_c(X)$ relate to measures on X by the classical Riesz's representation theorem. We shall only use the compact case, so we accordingly adapt the statement.

Fact 2.3 (Riesz's representation theorem)**.** *Let X be a compact Polish space. If $\Phi : C(X) \to \mathbb{C}$ is a normalized positive linear functional, there exists a unique outer Radon probability measure μ such that*

$$\Phi(f) = \int_X f d\mu$$

for every $f \in C(X)$.

2.2 Existence

Definition 2.4. For G a t.d.l.c. Polish group, a **left Haar measure** on G is a non-zero left-invariant outer Radon measure on G. The integral with respect to a left Haar measure is called the **left Haar integral**.

This section establishes the existence of left Haar measures.

Theorem 2.5 (Haar)**.** *Every t.d.l.c. Polish group admits a left Haar measure.*

We will first prove Theorem 2.5 for compact t.d.l.c. Polish groups and then upgrade to the t.d.l.c. Polish case.

Remark 2.6. The proofs of this section can be adapted to produce a non-zero right-invariant outer Radon measure on G, which is called a **right Haar measure**. We stress that in general one cannot produce a non-zero outer Radon measure that is simultaneously left and right invariant.

2.2.1 The compact case

For the compact case of Theorem 2.5, we obtain a rather stronger result; the reader is encouraged to identify the salient feature of compact t.d.l.c. groups which allows for this stronger result.

Theorem 2.7. *Every compact t.d.l.c. Polish group admits an outer Radon probability measure that is both left and right invariant.*

Our proof relies on the following technical lemma, Lemma 2.8. Let us for the moment assume Lemma 2.8 and prove Theorem 2.7.

Lemma 2.8. *For G a compact t.d.l.c. Polish group, there is a normalized positive linear functional Φ on $C(G)$ such that $\Phi(L_g(f)) = \Phi(f) = \Phi(R_g(f))$ for all $g \in G$ and $f \in C(G)$.*

Proof of Theorem 2.7. Let Φ be the normalized positive linear functional given by Lemma 2.8. Via Riesz's representation theorem, Φ defines an outer Radon probability measure μ on G. We argue that this measure is both left and right invariant.

As the proofs are the same, we argue that $\mu(gA) = \mu(A)$ for all $g \in G$ and $A \subseteq G$ Borel. Let us assume first that A is compact. Since the topology for G is given by cosets of clopen subgroups and A is compact, for any open set O with $A \subseteq O$, there is a clopen set O' such that $A \subseteq O' \subseteq O$. In particular, $\mu(O') \leq \mu(O)$. The outer regularity of μ now ensures that

$$\mu(A) = \inf\{\mu(O) \mid A \subseteq O \text{ with } O \text{ clopen}\}.$$

For any clopen set $O \subseteq G$, the indicator function 1_O is an element of $C(G)$. The left invariance of Φ ensures that

$$\mu(O) = \int_G L_g(1_O)d\mu = \int_G 1_{gO}d\mu = \mu(gO).$$

Since $\{gO \mid A \subset O \text{ and } O \text{ clopen}\}$ is exactly the collection of clopen sets containing gA, we deduce that

$$\begin{aligned}\mu(A) &= \inf\{\mu(O) \mid A \subseteq O \text{ with } O \text{ clopen}\} \\ &= \inf\{\mu(gO) \mid A \subseteq O \text{ with } O \text{ clopen}\} \\ &= \inf\{\mu(W) \mid gA \subseteq W \text{ with } W \text{ clopen}\} \\ &= \mu(gA).\end{aligned}$$

Hence, $\mu(A) = \mu(gA)$, so $\mu(A) = \mu(gA)$ for any $g \in G$ and $A \subseteq G$ compact.

Let us now consider an arbitrary Borel set A of G. By inner regularity,

$$\mu(A) = \sup\{\mu(K) \mid K \subseteq A \text{ with } K \text{ compact}\}.$$

Since $\{gK \mid K \subseteq A \text{ with } K \text{ compact}\}$ is exactly the collection of compact sets contained in gA, we conclude that

$$\begin{aligned}
\mu(A) &= \sup\{\mu(K) \mid K \subseteq A \text{ with } K \text{ compact}\} \\
&= \sup\{\mu(gK) \mid K \subseteq A \text{ with } K \text{ compact}\} \\
&= \sup\{\mu(W) \mid W \subseteq gA \text{ with } W \text{ compact}\} \\
&= \mu(gA).
\end{aligned}$$

Hence, $\mu(A) = \mu(gA)$ for any $g \in G$ and $A \subseteq G$ Borel. The measure μ thus satisfies Theorem 2.7. □

We now set about proving Lemma 2.8.

Remark 2.9. The strategy for our proof of Lemma 2.8 is to build the desired functional Φ via approximations. We will define a sequence of subspaces $\mathcal{F}_n \subseteq C(G)$ and a sequence of functionals $\varphi_n : \mathcal{F}_n \to \mathbb{C}$ which approximate the desired behavior of Φ. We will then argue that these functionals cohere to give Φ.

Fix G a compact t.d.l.c. Polish group. In view of Exercise 1.11, G admits a basis at 1 of compact open normal subgroups, so we may fix $(G_n)_{n\in\mathbb{N}}$ an $\subseteq$-decreasing basis at 1 of compact open normal subgroups of G. For each $n \in \mathbb{N}$, let $\mathcal{F}_n$ be the following subspace of $C(G)$:

$$\mathcal{F}_n := \{f \in \mathcal{C}(G) \mid \forall x \in G \text{ and } g \in G_n\ f(xg) = f(x)\}.$$

The set $\mathcal{F}_n$ is the vector space of all continuous functions from G to $\mathbb{C}$ which are right G_n-invariant.

The subspaces $\mathcal{F}_n$ are setwise stabilized by G under both the left and the right actions. That is to say, $R_g(\mathcal{F}_n) = \mathcal{F}_n = L_g(\mathcal{F}_n)$ for all $g \in G$. We verify the former equality, as the latter is similar but easier. Take $f \in \mathcal{F}_n$ and $g \in G$ and suppose that $x^{-1}y \in G_n$. We have $R_g(f)(x) = f(xg)$ and $R_g(f)(y) = f(yg)$. Additionally, $(xg)^{-1}yg = g^{-1}x^{-1}yg$. The element $x^{-1}y$ is in G_n, and since G_n is a normal subgroup, we infer that $g^{-1}x^{-1}yg \in G_n$. Therefore, $f(xg) = f(yg)$, and $R_g(f) \in \mathcal{F}_n$. It now follows that $R_g(\mathcal{F}_n) = \mathcal{F}_n$.

Fixing left coset representatives $b_1, \ldots, b_r$ for G_n in G, we obtain a positive linear functional φ_n on $\mathcal{F}_n$ defined by

$$\varphi_n(f) := \sum_{i=1}^{r} \frac{f(b_i)}{|G : G_n|}.$$

We stress that φ_n is independent of our choice of coset representatives. We argue that φ_n is invariant under the left and right actions of G.

Lemma 2.10. *For any $f \in \mathcal{F}_n$ and $g \in G$, $\varphi_n(L_g(f)) = \varphi_n(f) = \varphi_n(R_g(f))$.*

Proof. We only establish $\varphi_n(f) = \varphi_n(R_g(f))$; the other equality is similar but easier. Let $b_1, \ldots, b_r$ list coset representatives of G_n in G. Fix $g \in G$ and consider the set $b_i G_n g$. Since G_n is normal, we have $b_i g g^{-1} G_n g = b_i g G_n$. There is thus some j such that $b_i G_n g = b_j G_n$. We conclude there is a permutation σ of $\{1, \ldots, r\}$ such that $b_i G_n g = b_{\sigma(i)} G_n$. Hence, $b_i g = b_{\sigma(i)} u_i$ for some $u \in G_n$. We now see that

$$\begin{aligned} \varphi_n(R_g(f)) &= \textstyle\sum_{i=1}^{r} \frac{f(b_i g)}{|G:G_n|} \\ &= \textstyle\sum_{i=1}^{r} \frac{f(b_{\sigma(i)} u)}{|G:G_n|} \\ &= \textstyle\sum_{i=1}^{r} \frac{f(b_{\sigma(i)})}{|G:G_n|} \\ &= \varphi(f), \end{aligned}$$

where the penultimate line follows since f is constant on cosets of G_n. □

Our next lemma shows the sequence $(\varphi_n)_{n \in \mathbb{N}}$ has a coherence property.

Lemma 2.11. *For every $n \leq m$, the vector space $\mathcal{F}_n$ is a subspace of $\mathcal{F}_m$, and φ_n is the restriction of φ_m to $\mathcal{F}_n$.*

Proof. Take $f \in \mathcal{F}_n$. For any $x \in G$ and $g \in G_n$, we have $f(x) = f(xg)$. On the other hand, $G_n \geq G_m$, so a fortiori, $f(x) = f(xg)$ for any $x \in G$ and $g \in G_m$. Hence, $f \in \mathcal{F}_m$, and we deduce that $\mathcal{F}_n \leq \mathcal{F}_m$.

Let us now verify that $\varphi_m \upharpoonright_{\mathcal{F}_n} = \varphi_n$. We see that

$$|G : G_n||G_n : G_m| = |G : G_m|,$$

so

$$\frac{|G_n : G_m|}{|G : G_m|} = \frac{1}{|G : G_n|}.$$

Setting $k := |G_n : G_m|$ and $r := |G : G_n|$, let $a_1, \dots, a_k$ be left coset representatives for G_m in G_n and let $b_1, \dots, b_r$ be left coset representatives for G_n in G. Fixing $f \in \mathcal{F}_n$,

$$\varphi_m(f) = \sum_{i=1}^{r} \sum_{j=1}^{k} \frac{f(b_i a_j)}{|G : G_m|}.$$

The function f is constant on $b_i G_n$, so

$$\varphi_m(f) = \sum_{i=1}^{r} \frac{f(b_i a_1)k}{|G : G_m|} = \sum_{i=1}^{r} \frac{f(b_i)}{|G : G_n|}$$

Hence, $\varphi_m(f) = \varphi_n(f)$. $\square$

We now argue the subspaces $\mathcal{F}_n$ essentially exhaust $C(G)$. Let $\mathcal{F}_n^+$ be the positive functions in $\mathcal{F}_n$.

Lemma 2.12. *The union $\bigcup_{n\in\mathbb{N}} \mathcal{F}_n$ is dense in $C(G)$, and $\bigcup_{n\in\mathbb{N}} \mathcal{F}_n^+$ is dense in $C(G)^+$.*

Proof. Fix $\epsilon > 0$ and $f \in C(G)$. For every $x \in G$, the continuity of f ensures that there is an element G_i of our basis at the identity of G such that $\mathrm{diam}(f(xG_i)) < \epsilon$. Since G is compact and zero dimensional, we may find $x_1, \dots, x_p$ elements of G and natural numbers $n_1, \dots, n_p$ such that $(x_i G_{n_i})_{i=1}^p$ is a covering of G by disjoint open sets.

Fix $N \in \mathbb{N}$ such that

$$G_N \leq \bigcap_{i=1}^{p} G_{n_i},$$

and fix $r_x \in f(xG_N)$ for each left G_N-coset xG_N. We define $\tilde{f} \in \mathcal{F}_N$ by $\tilde{f}(x) := r_x$; that is, $\tilde{f}$ is the function that takes value r_x for any $y \in xG_N$. For $x \in G$, $x \in x_i G_{n_i}$ for some $1 \leq i \leq p$, so $xG_{n_i} = x_i G_{n_i}$. We infer that $f(x) \in f(x_i G_{n_i})$. On the other hand, xG_N is a subset of $x_i G_{n_i}$, as $G_N \leq G_{n_i}$, so $\tilde{f}(x) = r_x \in f(x_i G_{n_i})$. We now see that $|r_x - f(x)| < \epsilon$. Therefore, $\|\tilde{f} - f\| < \epsilon$, and we conclude that $\bigcup_{n\in\mathbb{N}} \mathcal{F}_n$ is dense in $C(G)$.

The second claim follows since if f positive, we can pick $r_x \geq 0$ for each coset $xG_N \in G/G_N$. $\square$

In view of Lemma 2.11, $\varphi_n(f) = \varphi_m(f)$ whenever $f \in \mathcal{F}_n \cap \mathcal{F}_m$. We may thus define a linear functional φ on $\mathcal{F} := \bigcup_{n\in\mathbb{N}} \mathcal{F}_n$ by $\varphi(f) := \varphi_n(f)$ for $f \in \mathcal{F}_n$. The linear functional φ is positive and normalized, and Lemma 2.12 tells us the domain of φ is dense in $C(X)$. We now argue that we may extend φ to $C(X)$. To avoid appealing to unproven facts about positive linear functionals, we verify that φ carries Cauchy sequences to Cauchy sequences.

Lemma 2.13. *If $(f_i)_{i\in\mathbb{N}}$ is a Cauchy sequence in $C(X)$ with $f_i \in \mathcal{F}$ for all $i \in \mathbb{N}$, then $(\varphi(f_i))_{i\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{C}$.*

Proof. Fix $\epsilon > 0$ and let N be such that $\|f_i - f_j\| < \epsilon$ for all $i, j \geq N$. Fix $i, j \geq N$ and let k be such that $f_i, f_j \in \mathcal{F}_k$. Letting $b_1, \ldots, b_m$ list left coset representatives for G_k in G,

$$\begin{array}{rcl} |\varphi(f_i) - \varphi(f_j)| & = & |\varphi_k(f_i) - \varphi_k(f_j)| \\ & = & \left|\sum_{l=1}^m \frac{f_i(b_l)-f_j(b_l)}{|G:G_k|}\right| \\ & \leq & \sum_{l=1}^m \frac{|f_i(b_l)-f_j(b_l)|}{|G:G_k|} \\ & < & \sum_{l=1}^m \frac{\epsilon}{|G:G_k|} \\ & = & \epsilon. \end{array}$$

We conclude that $(\varphi(f_i))_{i\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{C}$. □

Lemma 2.13 allows us to extend φ to a linear functional Φ on $C(G)$. For $f \in \mathcal{F}$, we define $\Phi(f) := \phi(f)$. For $f \in C(G)\backslash\mathcal{F}$, let $(f_i)_{i\in\mathbb{N}}$ be a sequence of functions from $\mathcal{F}$ that converges to f; Lemma 2.12 supplies these. We define $\Phi(f) := \lim_{i\in\mathbb{N}} \varphi(f_i)$. The latter limit exists by Lemma 2.13. The reader is encouraged to verify that this definition does not depend on the choice of the sequence $(f_i)_{i\in\mathbb{N}}$; see Exercise 2.1.

We now argue that Φ satisfies Lemma 2.8. It is immediate that Φ is normalized since $1_X \in \mathcal{F}$. To see that Φ is positive, let $f \in C(G)$ be a positive function. By Lemma 2.12, there is a sequence of positive functions $(f_i)_{i\in\mathbb{N}}$ with $f_i \in \mathcal{F}$ for all i which converges to f, so

$$\Phi(f) = \lim_{i\in\mathbb{N}} \varphi(f_i) = \lim_{i\in\mathbb{N}} \varphi_i(f_i) \geq 0.$$

The functional Φ is thus also positive.

Finally, let us verify that Φ is invariant under the left and right actions of G. Take $g \in G$ and $f \in C(X)$. If $f \in \mathcal{F}$, then

$$\Phi(L_g(f)) = \varphi(L_g(f)) = \varphi(f) = \varphi(R_g(f)) = \Phi(R_g(f))$$

by Lemma 2.10. Suppose that $f \in C(X) \setminus \mathcal{F}$ and fix $f_i \to f$ a convergent sequence such that $f_i \in \mathcal{F}$ for all i. Recall that $L_g(f_i) \to L_g(f)$ and $R_g(f_i) \to R_g(f)$. Applying the definition of Φ and Lemma 2.10, we deduce that

$$\Phi(L_g(f)) = \lim_{i\in\mathbb{N}} \varphi(L_g(f_i)) = \lim_{i\in\mathbb{N}} \varphi(f_i) = \lim_{i\in\mathbb{N}} \varphi(R_g(f_i)) = \Phi(R_g(f)),$$

hence $\Phi(L_g(f)) = \Phi(f) = \Phi(R_g(f))$.

2.2.2 The general case

The general case now follows by a clever trick, which allows one to glue together measures.

Lemma 2.14. *Suppose that X is a Polish space and $X = \bigsqcup_{i=1}^{\infty} U_i$ with each U_i a clopen subset of X. If each U_i admits an outer Radon probability measure ν_i, then the function $\mu : \mathcal{B} \to [0, \infty]$ by*

$$\mu(B) := \sum_{i=1}^{\infty} \nu_i(B \cap U_i),$$

is a non-trivial outer Radon measure on X.

Proof. That μ is a measure, we leave to the reader (Exercise 2.4). To see that μ is locally finite, for any $x \in X$, there is U_i such that $x \in U_i$. Hence, $\mu(U_i) = \nu_i(U_i) = 1$, so μ is locally finite.

Fix $B \in \mathcal{B}$. Outer regularity is immediate from the definition of μ if $\mu(B) = \infty$. Let us suppose that $\mu(B) < \infty$ and fix $\epsilon > 0$. For each $i \geq 1$, the outer regularity of ν_i supplies an open set $W_i \subseteq U_i$ such that $B \cap U_i \subseteq W_i$ and $\nu_i(W_i) - \nu_i(B \cap U_i) < \frac{\epsilon}{2^i}$. The set $W := \bigcup_{i=1}^{\infty} W_i$ is open in X. Furthermore,

$$\begin{array}{rcl} \mu(W) & = & \sum_{i=1}^{\infty} \nu_i(W_i) \\ & < & \sum_{i=1}^{\infty} \left(\nu_i(B \cap U_i) + \frac{\epsilon}{2^i}\right) \\ & = & \mu(B) + \epsilon. \end{array}$$

Since $B \subseteq W$, we deduce that $|\mu(W) - \mu(B)| < \epsilon$. The measure μ is thus outer regular.

For inner regularity, we may assume $\mu(B) < \infty$. Fix $\epsilon > 0$. Since the series $\sum_{i=1}^{\infty} \nu_i(B \cap U_i)$ converges, we may find N such that $\sum_{i=N+1}^{\infty} \nu_i(B \cap U_i) < \frac{\epsilon}{2}$. For each $1 \leq i \leq N$, we may find a compact set $K_i \leq B \cap U_i$

such that $\nu_i(B \cap U_i) - \mu_i(K_i) < \frac{\epsilon}{2N}$, by the inner regularity of ν_i. The set $K := \bigcup_{i=1}^{N} K_i$ is compact in X. Furthermore,

$$\begin{array}{rcl} \mu(B) & = & \sum_{i=1}^{\infty} \nu_i(B \cap U_i) \\ & < & \left(\sum_{i=1}^{N} \nu_i(K_i) + \frac{\epsilon}{2N} \right) + \sum_{i=N+1}^{\infty} \nu_i(B \cap U_i) \\ & = & \mu(W) + \epsilon. \end{array}$$

Since $K \subseteq B$, we deduce that $|\mu(B) - \mu(K)| < \epsilon$. The measure μ is thus inner regular. $\square$

For topological spaces X and Y, a function $f : X \to Y$ is **Borel measurable** if $f^{-1}(O)$ is in the Borel sigma algebra of X for any O open in Y. If X is additionally equipped with a Borel measure μ, then the **push forward** measure on Y via f, denoted by $f_*\mu$, is defined by

$$f_*\mu(A) := \mu(f^{-1}(A)).$$

If f is a homeomorphism and μ is an outer Radon measure, then $f_*\mu$ is also an outer Radon measure.

We are now prepared to prove the desired theorem.

Theorem 2.15 (Haar). *Every t.d.l.c. Polish group admits a left Haar measure.*

Proof. Let G be a t.d.l.c. Polish group and fix a compact open subgroup U of G. If U is of finite index in G, then G is a compact t.d.l.c. Polish group, so Theorem 2.7 supplies a left Haar measure on G. We thus suppose that U is of infinite index. Take $(g_i)_{i\in\mathbb{N}}$ a sequence of left coset representatives such that $G = \bigsqcup_{i\in\mathbb{N}} g_iU$.

Since U is a compact t.d.l.c. Polish group, Theorem 2.7 supplies a outer Radon probability measure ν on U which is both left and right invariant. Equip each g_iU with the pushforward measure $\nu_i := g_{i*}\nu$. That is to say, for $B \subseteq g_iU$ Borel,

$$\nu_i(B) = \nu(g_i^{-1}(B)).$$

The measure ν_i is an outer Radon measure on g_iU, since $g_i : U \to g_iU$ is a homeomorphism.

Applying Lemma 2.14, we obtain a non-trivial outer Radon measure μ on G such that for $B \subseteq G$ Borel,

$$\mu(B) := \sum_{i\in\mathbb{N}} \nu_i(B \cap g_iU).$$

Unpacking the definitions reveals that

$$\mu(B) = \sum_{i\in\mathbb{N}} \nu(g_i^{-1}(B \cap g_i U)) = \sum_{i\in\mathbb{N}} \nu(g_i^{-1}(B) \cap U).$$

We now argue that μ is left-invariant. For $g \in G$, there is a permutation σ of $\mathbb{N}$ such that $gg_iU = g_{\sigma(i)}U$, so in particular, $gg_i = g_{\sigma(i)}u$ for some $u \in U$. The element $g_{\sigma(i)}^{-1}gg_i$ is thus in U. For all Borel $B \subseteq X$, we now see that

$$\begin{aligned}
\mu(g^{-1}(B)) &= \textstyle\sum_{i\in\mathbb{N}} \nu((gg_i)^{-1}(B) \cap U) \\
&= \textstyle\sum_{i\in\mathbb{N}} \nu\left(g_{\sigma(i)}^{-1} gg_i\left((gg_i)^{-1}(B) \cap U\right)\right) \\
&= \textstyle\sum_{i\in\mathbb{N}} \nu(g_{\sigma(i)}^{-1}(B) \cap U) \\
&= \mu(B).
\end{aligned}$$

The second line follows from the left U-invariance of ν. We conclude that μ is left-invariant.

The measure μ is thus a left Haar measure on G. □

2.3 Uniqueness

The Haar measure enjoys a strong uniqueness property, which we will now prove.

Theorem 2.16 (Haar). *A left Haar measure on a t.d.l.c. Polish group is unique up to constant multiplies. That is to say, for any two left Haar measures μ_1 and μ_2, there is a non-zero real number c such that $\mu_1 = c\mu_2$.*

We argue that the Haar integral, and therefore the left Haar measure, is unique up to constant multiples. Several preliminary results are required.

Lemma 2.17. *Let μ be a left Haar measure on a t.d.l.c. Polish group G. Then*

(1) Every non-empty open set has strictly positive (possibly infinite) measure.

(2) Every compact set has finite measure.

(3) Every continuous positive function $f : G \to \mathbb{R}$ with $\int_G f d\mu = 0$ vanishes identically.

Proof. Exercise 2.14 □

For a measure space (X, μ), the collection of integrable functions $f : X \to \mathbb{C}$ is denoted by $L^1(X, \mu)$. When the space X is clear from context, we simply write $L^1(\mu)$.

Lemma 2.18. *Let μ be a left Haar measure on a t.d.l.c. Polish group G. For any $g \in G$ and $f \in L^1(\mu)$,*

$$\int_G L_g(f) d\mu = \int_G f d\mu.$$

Proof. Exercise 2.15. □

We in fact only use Lemma 2.18 for $f \in C_c(G)$, and this case follows by approximating a given $f \in C_c(G)$ by functions constant on left cosets of some compact open subgroup.

Lemma 2.19. *For G a t.d.l.c. Polish group, any function $f \in C_c(G)$ is uniformly continuous. That is to say, for every $\epsilon > 0$, there is $U \in \mathcal{U}(G)$ such that if $x^{-1}y \in U$ or $yx^{-1} \in U$, then $|f(x) - f(y)| < \epsilon$.*

Proof. Fix $\epsilon > 0$. We will find $U \in \mathcal{U}(G)$ such that if $x^{-1}y \in U$, then $|f(x) - f(y)| < \epsilon$; the other case is similar. One then simply intersects the compact open subgroups obtained in each argument to find a compact open subgroup that satisfies the lemma.

Fix $f \in C_c(G)$, let $U \in \mathcal{U}(G)$, and let $K := \overline{\operatorname{supp}(f)}$. For each $x \in G$, there is a compact open subgroup $V_x \leq U$ such that if $y \in xV_x$, then $|f(x) - v(y)| < \frac{\epsilon}{2}$. Since KU is compact, there are $x_1, \dots, x_n$ such that $KU = x_1 V_{x_1} \cup \dots \cup x_n V_{x_n}$. We claim $W := \bigcap_{i=1}^n V_{x_i}$ satisfies the lemma.

Suppose $x^{-1}y \in W$. If $x \notin KU$, then $y \notin K$, so $f(x) = f(y) = 0$. The desired inequality clearly holds in this case. Suppose $x \in KU$; say $x \in x_i V_{x_i}$. It is then the case that $y \in x_i V_{x_i}$, so

$$|f(x) - f(y)| \leq |f(x) - f(x_i)| + |f(x_i) - f(y)| < \frac{\epsilon}{2} + \frac{\epsilon}{2}.$$

proving the lemma. □

Lemma 2.20. *Let μ be a left Haar measure on G. For $f \in C_c(G)$, the function $\Psi_f : G \to \mathbb{C}$ defined by*

$$s \mapsto \int_G f(xs) d\mu(x)$$

is continuous.

Proof. Let us first show that Ψ_f is continuous at 1 for every $f \in C_c(G)$. That is, for every $\epsilon > 0$ there is $U \in \mathcal{U}(G)$ such that

$$\left| \int_G f(xs) - f(x) d\mu(x) \right| < \epsilon$$

for every $s \in U$.

Let $K := \operatorname{supp}(f)$, fix $\epsilon > 0$, and take $V \in \mathcal{U}(G)$. For $s \in V$, we see $\operatorname{supp}(R_s(f)) \subseteq KV$, and in view of Lemma 2.17, we observe that $0 < \mu(KV) < \infty$. Lemma 2.19 now supplies a compact open subgroup $W \leq V$ such that for all $s \in W$

$$|f(xs) - f(x)| < \frac{\epsilon}{\mu(KV)}.$$

Therefore,

$$\left| \int_G f(xs) - f(x) d\mu(x) \right| \leq \int_{KV} |f(xs) - f(x)| d\mu(x) \leq \frac{\epsilon}{\mu(KV)} \mu(KV).$$

verifying that Ψ_f is continuous at 1.

Fixing $f \in C_c(G)$, we argue that Ψ_f is in fact continuous. Take O open in $\mathbb{C}$ and fix r in $\Psi_f^{-1}(O)$. The function $R_r(f)$ is again an element of $C_c(G)$ and $\Psi_{R_r(f)}(1) \in O$. The preimage $(\Psi_{R_r(f)})^{-1}(O)$ thereby contains an open set L containing 1 such that $\Psi_{R_r(f)}(L) \subseteq O$. For any $zr \in Lr$,

$$\Psi_f(zr) = \int_G f(xzr) d\mu(x) = \int_G R_r(f(xz)) d\mu(x) = \Psi_{R_r(f)}(z),$$

so $\Psi_f(Lr) \subseteq O$, verifying that Ψ_f is continuous. $\square$

Our proof also requires the classical Fubini–Tonelli theorem. A measure space (X, μ) is called **sigma finite** if $X = \bigcup_{n \in \mathbb{N}} W_n$ such that each W_n is measurable with $\mu(W_n) < \infty$. It is an easy exercise to see that any Haar measure is sigma finite (Exercise 2.12).

Fact 2.21 (Fubini–Tonelli)**.** *Suppose that (X, μ) and (Y, ν) are sigma finite measure spaces and take $f \in L^1(\mu \times \nu)$, the space of integrable functions $f : X \times Y \to \mathbb{C}$ where $X \times Y$ is equipped with the product measure. Then,*

$$\begin{aligned} \int_{X \times Y} f(x,y) d(\mu \times \nu) &= \int_X \left(\int_Y f(x,y) d\nu(y) \right) d\mu(x) \\ &= \int_Y \left(\int_X f(x,y) d\mu(x) \right) d\nu(y). \end{aligned}$$

Proof of Theorem 2.16. Suppose ν and μ are two left Haar measures on G. For $f \in C_c(G)$ with $I_\mu(f) := \int_G f d\mu \neq 0$, set

$$D_f(s) := \frac{1}{I_\mu(f)} \int_G f(ts) d\nu(t).$$

By Lemma 2.20, $D_f(s)$ is a continuous function of s

Take $g \in C_c(G)$. Via the Fubini–Tonelli theorem,

$$I_\mu(f) I_\nu(g) = \int_G \left(\int_G f(s) g(t) d\nu(t) \right) d\mu(s).$$

Lemma 2.18 ensures that

$$\begin{array}{rcl} \int_G \int_G f(s)g(t)d\nu(t)d\mu(s) & = & \int_G \int_G L_s(f(s)g(t))d\nu(t)d\mu(s) \\ & = & \int_G \int_G f(s)g(s^{-1}t)d\nu(t)d\mu(s) \end{array}$$

A second application of the Fubini–Tonelli theorem yields

$$= \int_G \int_G f(s)g(s^{-1}t)d\nu(t)d\mu(s) = \int_G \left(\int_G f(s)g(s^{-1}t)d\mu(s) \right) d\nu(t),$$

and appealing again to invariance,

$$\begin{array}{rcl} \int_G \int_G f(s)g(s^{-1}t)d\mu(s)d\nu(t) & = & \int_G \int_G L_{t^{-1}}(f(s)g(s^{-1}t))d\mu(s)d\nu(t) \\ & = & \int_G \int_G f(ts)g(s^{-1})d\nu(t)d\mu(s) \\ & = & I_\mu(f) \int_G D_f(s)g(s^{-1})d\mu(s). \end{array}$$

We have now demonstrated that

$$I_\mu(f) I_\nu(g) = I_\mu(f) \int_G D_f(s) g(s^{-1}) d\mu(s),$$

and as $I_\mu(f) \neq 0$, $I_\nu(g) = \int_G D_f(s)g(s^{-1})d\mu(s)$.

For any other function $f' \in C_c(G)$ with $I_\mu(f') \neq 0$, it is thus the case that

$$\int_G (D_f(s) - D_{f'}(s))\, g(s^{-1}) d\mu(s) = 0.$$

As this equality holds for any $g \in C_c(G)$, let us replace g with

$$\tilde{g}(s) := |g(s)|^2 \overline{(D_f(s^{-1}) - D_{f'}(s^{-1}))},$$

where $\overline{(D_f(s^{-1}) - D_{f'}(s^{-1}))}$ denotes the complex conjugate. Observe that $\tilde{g} \in C_c(G)$, via Lemma 2.20. We now see that

$$0 = \int_G (D_f(s) - D_{f'}(s))\, \tilde{g}(s^{-1}) d\mu(s) = \int_G |D_f(s) - D_{f'}(s)|^2 |g(s^{-1})|^2 d\mu(s).$$

In view of Lemma 2.17, $(D_f(s) - D_{f'}(s))g(s^{-1}) = 0$ for all s. Since $g(x)$ is arbitrary, $D_f(s) = D'_f(s)$ for all f and f' in $C_c(G)$ with non-zero integral, so $D_f(s) \equiv D$ for some constant D independent of f.

Recalling the definition of $D_f(s)$, we see that

$$D = D_f(s) = \frac{1}{I_\mu(f)} \int_G f(ts) d\nu(t).$$

Taking $s = 1$, we deduce that

$$DI_\mu(f) = \int_G f(t) d\nu(t) = I_\nu(f).$$

Hence, $D\mu = \nu$ verifying that the Haar measure is unique up to constant multiples. □

2.4 The modular function

Let G be a t.d.l.c. Polish group with left Haar measure μ and $x \in G$. We may produce a new left Haar measure μ_x by defining $\mu_x(A) := \mu(Ax)$. The uniqueness of the Haar measure implies there is $\Delta(x) > 0$ such that $\mu_x = \Delta(x)\mu$.

Definition 2.22. The map $\Delta : G \to \mathbb{R}_{>0}$ defined above is called the **modular function** for G. If $\Delta \equiv 1$, then G is said to be **unimodular**.

A priori, it may seem the modular function depends on the choice of Haar measure μ, but this is not the case.

Lemma 2.23. *Let G be a t.d.l.c. Polish group and μ_1 and μ_2 be left Haar measures on G. Taking Δ_1 and Δ_2 to be the modular functions defined in terms of μ_1 and μ_2, respectively, $\Delta_1 = \Delta_2$.*

Proof. Since the Haar measure is unique up to constant multiples, there is a positive real number D such that $\mu_1 = D\mu_2$. For all $x \in G$ and $A \subseteq G$ measurable, we have

$$D\mu_2(Ax) = \mu_1(Ax) = \Delta_1(x)\mu_1(A) = \Delta_1(x)D\mu_2(A).$$

Hence, $\mu_2(Ax) = \Delta_1(x)\mu_2(A)$ for all $x \in G$ and $A \subseteq G$ measurable, so $\Delta_2 = \Delta_1$. $\square$

The modular function is additionally a continuous homomorphism.

Proposition 2.24. *For G a t.d.l.c. Polish group, the modular function $\Delta : G \to \mathbb{R}_{>0}$ is a continuous group homomorphism.*

Proof. We first argue Δ is a group homomorphism. Fixing $U \in \mathcal{U}(G)$, $x, y \in G$, and μ a left Haar measure,

$$\begin{array}{rcl}\Delta(xy)\mu(U) & = & \mu(Uxy) \\ & = & \Delta(y)\mu(Ux) \\ & = & \Delta(x)\Delta(y)\mu(U).\end{array}$$

Since $0 < \mu(U) < \infty$, $\Delta(xy) = \Delta(x)\Delta(y)$ verifying that Δ is a group homomorphism.

To see that Δ is continuous, we have only to check continuity at 1, since Δ is a group homomorphism. Fix W a compact open subgroup. By the uniqueness of Haar measure, μ restricted to W is a Haar measure on W. Theorem 2.7 therefore implies that $\mu(Aw) = \mu(A)$ for all measurable $A \subseteq W$ and $w \in w$. Hence, $\Delta(w) = 1$ for all $w \in W$, so Δ is continuous at 1. $\square$

For G a t.d.l.c. Polish group and $H \leq G$ a closed subgroup, the left Haar measure μ_H on H and the left Haar measure μ_G on G are often very different. For instance, unless H is open in G, $\mu_G(H) = 0$; see Exercise 2.18. The respective modular functions can also differ. We shall write Δ_H to denote the modular function of H and Δ_G to denote that of G.

2.5 Quotient integral formula

Given a t.d.l.c. Polish group G and a closed subgroup $H \leq G$, one obtains a locally compact space G/H on which G acts continuously by left multiplication. Under mild conditions on G and H, G/H furthermore admits an outer Radon measure that is invariant under the action of G.

Definition 2.25. Suppose that G is a group acting on a measure space (X, μ). We say that μ is **invariant** under the action of G if $\mu(A) = \mu(g(A))$ for all $g \in G$ and measurable $A \subseteq X$.

Theorem 2.26 (Quotient integral formula)**.** *For G a t.d.l.c. Polish group and $H \leq G$ a closed subgroup, G/H admits an invariant non-zero outer Radon measure ν if and only if $\Delta_G \restriction_H = \Delta_H$. If the equivalent conditions hold, then the following additionally hold:*

(1) The measure ν is unique up to constant multiples.

(2) Given μ_G and μ_H left Haar measures on G and H respective, there is a unique choice for ν such that for every $f \in C_c(G)$,

$$\int_G f(x) d\mu_G(x) = \int_{G/H} \int_H f(xh) d\mu_H(h) d\nu(x).$$

*This relationship is called the **quotient integral formula**.*

Proof. to be added □

2.6 Lattices

Definition 2.27. For G a t.d.l.c. Polish group and Γ a discrete subgroup, we say that Γ is a **lattice** if G/Γ admits an invariant outer Radon probability measure.

Proposition 2.28. *Let Γ be a discrete subgroup of a t.d.l.c. Polish group G.*

1. *If Γ is cococompact in G, then Γ is a lattice.*

2. *If Γ is a lattice, then G is unimodular.*

Proof. to be added □

Definition 2.29. For G a Polish group and H a closed subgroup of G, we say that $\Omega \subseteq G$ is a **fundamental domain** for H in G if $|\Omega \cap gH| = 1$ for all $g \in G$. Alternatively, Ω contains exactly one element of each left coset of H in G.

Lemma 2.30. *For G a t.d.l.c. Polish group, if $\Gamma \leq G$ is discrete, then there exists a Borel fundamental domain for Γ in G.*

Theorem 2.31. *Let G be a t.d.l.c. Polish group with $\Gamma \leq G$ a discrete subgroup.*

(1) If Γ is a lattice in G, then every Borel fundamental domain for Γ has finite Haar measure.

(2) If G is unimodular and there exists a Borel fundamental domain for Γ in G with finite Haar measure, then Γ is a lattice in G.

Proof. to be added □

Notes

For a thorough treatment of measure theory as well as the proof of the Riesz representation theorem, we direct the reader to [8].

The existence proof for the Haar measure was explained to us by F. Le Maître; we thank Le Maître for his elegant contribution. Our approach to the uniqness of Haar measure and the quotient integral formula follows that of A. Deitmar and S. Echterhoff in [7]. The existence and uniqueness results for left or right Haar measure indeed hold for any locally compact group. The interested reader is directed to [7] or [9].

A more in depth discussion of lattices may be found in cite [3, Appendix B].

2.7 Exercises

Topology and measure theory

Exercise 2.1. Let X and Y be metric spaces with Y a compete metric space. Suppose that $Z \subseteq X$ is dense and $f : Z \to Y$ is such that for any Cauchy sequence $(x_n)_{n\in\mathbb{N}}$ in X, the image $(f(x_n))_{n\in\mathbb{N}}$ is a Cauchy sequence in Y. Show there is a unique continuous function $\tilde{f} : X \to Y$ such that the restriction of $\tilde{f}$ to Z is f.

Exercise 2.2. Suppose that X and Y are Polish spaces with $f : X \to Y$ a Borel measurable function. Show $f^{-1}(A)$ is Borel for any Borel set A of Y.

Exercise 2.3. Let X and Y be topological spaces with $f : X \to Y$ a homeomorphism. Show that if X is equipped with an outer Radon measure μ, then $f_*\mu$ is an outer Radon measure on Y.

Exercise 2.4. Verify the function μ defined in Lemma 2.14 is indeed a measure.

Exercise 2.5. Let X be a Polish space and equip $C_c(X)$ with the norm topology. Show $C_c(X)$ is a topological vector space. That is, show the vector space operations are continuous with respect to the norm topology.

Exercise 2.6. For G a Polish group, suppose that $f_i \in C_c(G)$ converges to some $f \in C_c(G)$ in the uniform topology. Show $L_g(f_i) \to L_g(f)$ and $R_g(f_i) \to R_g(f)$ for any $g \in G$.

Exercise 2.7. Let X be a Polish space and Φ be a positive linear functional on $C_c(X)$. Show Φ is continuous.

Haar measure

Exercise 2.8. Let G be a t.d.l.c. Polish group and $g \in C_c(G)$. Show there is $h \in C_c^+(G)$ such that $h \equiv 1$ on $\mathrm{supp}(g)$.

Exercise 2.9. Let G be a t.d.l.c. Polish group. Show that if G admits a Haar measure that is both left and right invariant, then every Haar measure is both left and right invariant.

Exercise 2.10. Let G be a t.d.l.c. Polish group. Show that if G is abelian or compact, then $\Delta \equiv 1$.

Exercise 2.11. Let G be a t.d.l.c. Polish group. Show G has finite Haar measure if and only if G is compact.

Exercise 2.12. Show any Haar measure is sigma finite.

Exercise 2.13. Let G be a profinite Polish group and $H \trianglelefteq_o G$. Show

$$F := \{f : G \to \mathcal{C} \mid f \text{ is continuous and} f(x) = f(xh) \text{ for all } h \in H\}$$

is isomorphic as a vector space to $L := \{f : G/H \to \mathbb{C}\}$.

Exercise 2.14. Prove Lemma 2.17.

Exercise 2.15. Let G be a t.d.l.c. Polish group and μ the left Haar measure on G. For $g \in G$ and $f \in L^1(G)$, show the following:

(a) $R_g(f)$ and $L_g(f)$ are elements of $L^1(G)$.

(b) $\int_G R_g(f(x))d\mu(x) = \Delta(g^{-1}) \int_G f(x)d\mu(x)$ where Δ is the modular function.

(c) $\int_G L_g(f(x))d\mu(x) = \int_G f(x)d\mu(x)$. This shows the left Haar integral is left-invariant as a linear operator on $L^1(G)$. In particular, the left Haar integral is a left-invariant linear operator on $C_c(G)$.

Hint: First consider simple functions and then approximate.

Exercise 2.16. Let G be a t.d.l.c. Polish group with left Haar measure μ. Show the following are equivalent:

(1) There is $x \in G$ such that $\mu(\{x\}) > 0$.

(2) The set $\{1\}$ has positive measure.

(3) The Haar measure is a multiple of counting measure.

(4) G is a discrete group.

Exercise 2.17. Let G be a t.d.l.c. Polish group with left Haar measure μ. Show the modular function Δ only takes rational values. Argue further that $G/\ker(\Delta)$ is a discrete abelian group.

Exercise 2.18 (weil)**.** Let G be a t.d.l.c. Polish group with left Haar measure μ. Show if $A \subseteq G$ is measurable with $\mu(A) > 0$, then AA^{-1} contains a neighborhood of 1. Use this to prove that if $H \leq G$ is a subgroup with positive measure, then H is open.

HINT: use inner and outer regularity.

Lattices

Exercise 2.19. Let Γ be a cocompact lattice in a t.d.l.c. Polish group G. Show $\Gamma \cap C_G(\gamma)$ is a lattice in $C_G(\gamma)$ for any $\gamma \in \Gamma$.

Chapter 3

Geometric Structure

For G a finitely generated group with X a finite symmetric generating set for G, the **Cayley graph** for G with respect to X, denoted by $\mathrm{Cay}(G, X)$, is defined as follows: $V\mathrm{Cay}(G, X) := G$ and

$$E\mathrm{Cay}(G, X) := \{\{g, gx\} \mid g \in G \text{ and } x \in X \setminus \{1\}\}.$$

It is easy to see that $\mathrm{Cay}(G, X)$ is a locally finite connected graph on which G acts vertex transitively and freely. What is more striking is that $\mathrm{Cay}(G, X)$ is unique up to *quasi-isometry*; we will define and explore this notion of equivalence below. The uniqueness of $\mathrm{Cay}(G, X)$ up to quasi-isometry allows one to produce new group invariants that are *geometric* and therefore allows one to study a group as a geometric object.

Studying the geometric properties of finitely generated groups has been fruitful, so one naturally wishes to cast t.d.l.c. groups as geometric objects. We will see that the compact open subgroups given by van Dantzig's theorem allow us to generalize the Cayley graph to compactly generated t.d.l.c. groups.

3.1 The Cayley–Abels graph

Definition 3.1. For a t.d.l.c. group G, a locally finite connected graph Γ on which G acts vertex transitively with compact open vertex stabilizers is called a **Cayley–Abels graph** for G.

Immediately, we see that the existence of a Cayley–Abels graph ensures compact generation.

Proposition 3.2. *If a t.d.l.c. group admits a Cayley–Abels graph, then it is compactly generated.*

Proof. Let G be a t.d.l.c. group with a Cayley–Abels graph Γ. Fix $v \in V\Gamma$ and let $G_{(v)}$ denote the stabilizer of the vertex v in G. The subgroup $G_{(v)}$ is compact and open by the definition of a Cayley–Abels graph.

Since Γ is locally finite and G acts vertex transitively, there are $g_1, \dots, g_n \in G$ such that

$$\{g_1.(v), \dots, g_n(v)\}$$

lists the neighbors of v. Let $F := \langle g_1, \dots, g_n \rangle$. We now argue by induction on $d_\Gamma(v, w)$ for the claim that there is $\gamma \in F$ such that $\gamma(v) = w$.

The base case, $d_\Gamma(v, w) = 0$, is obvious. Suppose the hypothesis holds up to k and $d_\Gamma(v, w) = k+1$. Let $v, u_1, \dots, u_k, w$ be a geodesic from v to w. By the induction hypothesis, $u_k = \gamma(v)$ for some $\gamma \in F$. Therefore, $\gamma^{-1}(u_k) = v$, and $\gamma^{-1}(w) = g_i(v)$ for some $1 \leq i \leq n$. We conclude $\gamma g_i(v) = w$ verifying the induction hypothesis.

For all $g \in G$ there is $\gamma \in F$ such that $g(v) = \gamma(v)$, so $\gamma^{-1} g \in G_{(v)}$. We conclude that $G = FG_{(v)}$, so $G = \langle g_1, \dots, g_n, G_{(v)} \rangle$ and is compactly generated. $\square$

Much less obviously, every compactly generated t.d.l.c. group admits a Cayley–Abels graph. Our preliminary lemma isolates the one ball of such a graph.

Lemma 3.3. *Suppose that G is a compactly generated t.d.l.c. group, X is a compact generating set, and $U \in \mathcal{U}(G)$. Then the following hold:*

(1) There is a finite symmetric set $A \subseteq G$ containing 1 such that $X \subseteq AU$ and $UAU = AU$.

(2) For any finite symmetric set A containing 1 with $X \subseteq AU$ and $UAU = AU$, it is the case that $G = \langle A \rangle U$.

Proof. Since $\{xU \mid x \in X\}$ is an open cover of X, we may find B a finite symmetric set containing 1 such that $X \subseteq BU$. On the other hand, UB is a compact set, so there is a finite symmetric set A containing 1 such that $B \subseteq A \subseteq UBU$ and $UB \subseteq AU$. We conclude that $UAU = UBU \subseteq AUU = AU$, so $UAU = AU$, verifying the first claim.

For the second claim, let us argue by induction on n that $(UAU)^n = A^nU$ for all $n \geq 1$. The base case is given by our hypotheses. Supposing that $(UAU)^n = A^nU$, we see that

$$(UAU)^{n+1} = (UAU)^n UAU = A^n UUAU = A^n UAU = A^{n+1}U,$$

completing the induction. Since UAU contains X and is symmetric, it now follows that

$$G = \langle UAU \rangle = \bigcup_{n\geq 1}(UAU)^n = \bigcup_{n\geq 1} A^nU = \langle A \rangle U. \qquad \square$$

Remark 3.4. The factorization produced in Lemma 3.3 need not have any algebraic content. The group G need not be an amalgamated free product or semidirect product of $\langle A \rangle$ and U.

Lemma 3.3 suggests a construction of a Cayley–Abels graph. The vertices ought to be left cosets of U, and the set AU, where A is as found in Lemma 3.3, of left cosets of U forms the one ball around coset U. Our next theorem fills in the details of this intuition; the reader is encouraged to compare the construction of the Cayley–Abels graph below with the construction of the classical Cayley graph discussed above.

Theorem 3.5 (Abels). *For G a compactly generated t.d.l.c. group and U a compact open subgroup of G, there is a Cayley-Abels graph Γ for G such that $V\Gamma = G/U$. In particular, there is $v \in V\Gamma$ such that $G_{(v)} = U$.*

Proof. Applying Lemma 3.3, there is a finite symmetric set A which contains 1 such that $UAU = AU$ and $G = \langle A \rangle U$. We define the graph Γ by $V\Gamma := G/U$ and

$$E\Gamma := \{\{gU, gaU\} \mid g \in G \text{ and } a \in A \setminus \{1\}\}.$$

We argue Γ satisfies the theorem. It is clear that G acts vertex transitively on Γ with compact open vertex stabilizers; the vertex stabilizers are conjugates of U. It remains to show that Γ is connected and locally finite. For connectivity, take $gU \in V\Gamma$. Lemma 3.3 ensures that $G = \langle A \rangle U$, so we may write $g = a_1 ... a_n u$ for $a_1, \dots, a_n$ elements of A and $u \in U$. Thus,

$$U, a_1U, a_1a_2U, \dots, gU$$

is a path in Γ connecting U to gU. We deduce that Γ is connected.

For local finiteness, it suffices to show $B_1(U) = \{aU \mid a \in A\} = AU$, since G acts on Γ vertex transitively. If $\{kU, kaU\}$ is an edge in Γ with U as an end point, then either $k \in U$ or $kaU = U$. In the former case,

$$kaU \in UAU = AU.$$

For the latter, $k = ua^{-1}$ for some $u \in U$, so $kU = ua^{-1}U$. We conclude that $kU \in UAU = AU$, since A is symmetric. In either case, the edge $\{kU, kaU\}$ is of the form $\{U, a'U\}$ for some $a' \in A$. Hence, $B_1(U) = AU$. We conclude that Γ is locally finite. □

Corollary 3.6 (Abels)**.** *A t.d.l.c. group admits a Cayley–Abels graph if and only if it is compactly generated.*

Proof. The forward implication is given by Proposition 3.2. The reverse is given by Theorem 3.5. □

Remark 3.7. As soon as a compactly generated t.d.l.c. group is non-discrete, the action on a Cayley–Abels graph is *never* free. That is to say, the action always has non-trivial vertex stabilizers. We shall see that these large, but compact stabilizers play an important role in the structure of compactly generated t.d.l.c. groups.

For the moment, let us denote the graph built in the proof of Theorem 3.5 by $\Gamma_{A,U}$. That is, U is a compact open subgroup of G and A is a finite symmetric set such that AU is a generating set for G and $UAU = AU$. The graph $\Gamma_{A,U}$ is then defined by $V\Gamma_{A,U} := G/U$ and

$$E\Gamma_{A,U} := \{\{gU, gaU\} \mid g \in G \text{ and } a \in A \setminus \{1\}\}.$$

Our next lemmas show that every Cayley-Abels graph is of the form $\Gamma_{A,U}$ for some finite symmetric A and compact open subgroup U.

Lemma 3.8. *Suppose that G is a compactly generated t.d.l.c. group and Γ is a Cayley-Abels graph for G. Fix $v \in V\Gamma$, set $U := G_{(v)}$, and let $B \subseteq G$ be finite containing* 1 *such that $B(v) = B_1(v)$. Setting $A := B \cup B^{-1}$, the following hold:*

(1) $A(v) = B_1(v)$;

(2) $UAU = AU$; and

(3) $G = \langle A \rangle U$.

Proof. For any $b \in B$, the edge $\{b(v), v\}$ is an edge of Γ. Therefore, $\{v, b^{-1}(v)\}$ is an edge in Γ, so $b^{-1}(v) \in B_1(v)$. We conclude that $A(v) = B_1(v)$, verifying (1).

Taking $a \in A$, the vertex $a(v)$ is a member of $B_1(v)$, and as U is the stabilizer of v, ua is also a member of $B_1(v)$. Thus, $ua(v) = a'(v)$ for some $a' \in A$, so $uaU = a'U$. We deduce that $UAU = AU$, verifying (2).

The proof of Proposition 3.2 shows that AU is a generating set for G. Applying Lemma 3.3, we obtain (3). □

Lemma 3.8 shows that for A and U as above, we may form the Cayley–Abels graph $\Gamma_{A,U}$. The next lemma shows this is precisely the same graph we started with.

Lemma 3.9. *Suppose that G is a compactly generated t.d.l.c. group and Γ is a Cayley-Abels graph for G. Fix $v \in V\Gamma$, set $U := G_{(v)}$, let $B \subseteq G$ be finite containing* 1 *such that $B(v) = B_1(v)$, and put $A := B \cup B^{-1}$. Then there is a G-equivariant graph isomorphism $\psi : \Gamma \to \Gamma_{A,U}$.*

Proof. In view of Lemma 3.8, we may form the graph $\Gamma_{A,U}$.

For each $w \in V\Gamma$, fix $g_w \in G$ such that $g_w(v) = w$. We obtain a bijection $\psi : \Gamma \to \Gamma_{A,U}$ by $w \mapsto g_w U$. The reader verifies that $\psi(g(w)) = g\psi(w)$ for all $g \in G$ and $w \in V\Gamma$; that is, the map ψ is G-equivariant.

Let $\{g_w(v), g_u(v)\} \in E\Gamma$; the case that $\{g_w(v), g_u(v)\} \notin E\Gamma$ is similar. We see that $\{v, g_w^{-1}g_u(v)\} \in E\Gamma$, so $g_w^{-1}g_u = bu$ for some $b \in B$ and $u \in U$. Hence, $\{g_wU, g_uU\} = \{g_uU, g_ubU\}$, so $\{g_wU, g_uU\} \in E\Gamma_{A,U}$. We conclude that ψ is such that $\{v, w\} \in E\Gamma$ if and only if $\{\psi(v), \psi(w)\} \in \Gamma_{A,U}$, so ψ is a graph isomorphism. □

In view of Lemma 3.9, the notation $\Gamma_{A,U}$ is superfluous, so we discard it.

3.2 Cayley-Abels representations

Given a compactly generated t.d.l.c. group G and a Cayley–Abels graph Γ, we obtain a representation $\psi : G \to \mathrm{Aut}(\Gamma)$ induced from the action of G on Γ. This representation is called the **Cayley-Abels representation** of G. Recalling that $\mathrm{Aut}(\Gamma)$ is a t.d.l.c. group in its own right, one naturally wishes to understand this representation better.

An important concept here, and generally in the study of locally compact groups, is that of a cocompact subgroup.

Lemma 3.10. *For G a topological group with H a closed subgroup, the space of left cosets G/H is compact if and only if the space of right cosets $H\backslash G$ is compact.*

Proof. Exercise 3.2 □

Definition 3.11. For G a topological group with H a closed subgroup of G, we say that H is **cocompact** in G if the quotient space G/H, equivalently $H\backslash G$, equipped with the quotient topology is compact.

Cayley–Abels graphs allow us to easily identify cocompact subgroups.

Lemma 3.12. *Let G be a compactly generated t.d.l.c. group, H be a closed subgroup of G, and Γ be a Cayley–Abels graph for G. Then H is cocompact in G if and only if H has finitely many orbits on $V\Gamma$.*

Proof. Suppose first that H is cocompact in G. The space of right cosets $H\backslash G$ is then compact. Fix $w \in V\Gamma$ and let $U := G_{(w)}$. The collection $\{HgU \mid g \in G\}$ forms an open cover of $H\backslash G$, so there is a finite subcover. We may thus find $X := g_1U \cup \dots \cup g_nU$ such that $HX = G$. The action of G on $V\Gamma$ is transitive, so $G(w) = V\Gamma$. Hence, $HX(w) = \cup_{i=1}^{n} H(g_i(w)) = V\Gamma$. We infer that H has finitely many orbits on $V\Gamma$.

Conversely, suppose that H has finitely many orbits on $V\Gamma$. Let $v_1, \dots, v_n$ be representatives for the orbits of H on $V\Gamma$, fix $w \in V\Gamma$, and fix $g_1, \dots, g_n$ in G such that $g_i(w) = v_i$ for $1 \leq i \leq n$. For $g \in G$, the vertex $g(w)$ is in some orbit of H, so there is $h \in H$ such that $hg(w) = v_i$ for some $1 \leq i \leq n$. Thus $g_i^{-1}hg \in G_{(w)}$. Setting $F := \{g_1, \dots, g_n\}$, it follows that $G = HFG_{(w)}$. As $FG_{(v)}$ is compact, the set of right cosets $H\backslash G$ is compact. Hence, H is cocompact in G. □

We are now prepared to give some insight into the representation induced by a Cayley–Abels graph.

Theorem 3.13. *Let G be a compactly generated t.d.l.c. group and Γ a Cayley–Abels graph for G. Then the induced homomorphism $\psi : G \to \mathrm{Aut}(\Gamma)$ is a continuous and closed map, $\psi(G)$ is cocompact in $\mathrm{Aut}(\Gamma)$, and $\ker(\psi)$ is compact.*

Proof. Set $H := \mathrm{Aut}(\Gamma)$. Since ψ is a homomorphism, it suffices to show that ψ is continuous at 1. A basis at the identity in H consists of pointwise vertex stabilizers $H_{(F)}$ where F is a finite set of vertices. We see that $\psi^{-1}(H_{(F)}) = G_{(F)}$, and as vertex stabilizers in G are open in G, we conclude that $G_{(F)}$ is open. Hence, ψ is continuous.

To see that ψ is closed, fix $A \subseteq G$ closed and say that $\psi(a_i) \to h \in H$. Fixing $w \in V\Gamma$, there is N such that $\psi(a_i^{-1})\psi(a_j) \in H_{(w)}$ for all $i, j \geq N$, so $a_i^{-1}a_j \in G_{(w)}$ for all $i, j \geq N$. Fix $i \geq N$. As $G_{(w)}$ is compact and $a_i^{-1}a_j \in G_{(w)}$ for all $j \geq N$, there is a convergent subsequence $a_i^{-1}a_{j_k} \to b$, and it follows that the subsequence a_{j_k} converges to some $a \in A$. As ψ is continuous, we deduce that $\psi(a) = h$, so ψ is a closed map.

The image $\psi(G)$ is now a closed subgroup of H that acts vertex transitively on $V\Gamma$. Applying Lemma 3.12, we conclude that $\psi(G)$ is cocompact in H. The final claim is immediate since vertex stabilizers are compact. $\square$

Remark 3.14. Theorem 3.13 shows that compactly generated t.d.l.c. groups are very close to being Polish groups. Every compactly generated t.d.l.c. group is compact-by-Polish. This observation shows our restriction to t.d.l.c. Polish groups loses little generality.

Let us note a further observation on cocompact subgroups; the proof illustrates a useful technique for manipulating Cayley–Abels graphs.

Proposition 3.15. *For G a compactly generated t.d.l.c. group, if H is a closed and cocompact subgroup of G, then H is compactly generated.*

Proof. Fix Γ a Cayley–Abels graph for G. In view of Lemma 3.12, the subgroup H has finitely many orbits on $V\Gamma$. Let $v_1, \dots, v_n$ list representatives of the orbits of H on $V\Gamma$. If H acts transitively on $V\Gamma$, then Γ is a Cayley–Abels graph for H, so H is compactly generated by Proposition 3.2. We thus suppose that H is intransitive and define $m := \mathrm{diam}(\{v_1, \dots, v_n\})$. Note that $m \geq 1$.

Letting O be the orbit of v_1 under the action of H, we argue that for any two $r, s \in O$, there is a sequence of vertices $r := u_1, \dots, u_n =: s$ in O such that $d_\Gamma(u_i, u_{i+1}) \leq 2m + 1$. Let $r = w_1, \dots, w_n = s$ be a path from r to s. We argue by induction on $i < n$ that we can find vertices $u_1, \dots, u_i$ in O such that $r = u_1$, $d_\Gamma(u_j, u_{j+1}) \leq 2m + 1$ for $j < i$, and $d(u_i, w_{i+1}) \leq m + 1$.

For the base case, $i = 1$, we simply set $u_1 = w_1$. Suppose we have found $u_1, \dots, u_i$. If $w_{i+1} \in O$, then we set $u_{i+1} = w_{i+1}$, and we are done. Let us assume that $w_{i+1} \notin O$, so in particular, $i+1 < n$. Suppose that w_{i+1} is in the orbit of v_k for some $k \neq 1$. Let $h \in H$ be such that $h(v_k) = w_{i+1}$ and set $u_{i+1} := h(v_1)$. Clearly, $u_{i+1} \in O$. Furthermore, $d(u_{i+1}, w_{i+1}) \leq m$, so we deduce that

$$d(u_i, u_{i+1}) \leq d(u_i, w_{i+1}) + d(w_{i+1}, u_{i+1}) \leq m + 1 + m = 2m + 1.$$

Finally,

$$d(u_{i+1}, w_{i+2}) \leq d(u_{i+1}, w_{i+1}) + d(w_{i+1}, w_{i+2}) \leq m + 1,$$

completing the induction.

We now build a new graph Δ as follows: $V\Delta = V\Gamma$ and

$$E\Delta := \{\{v, w\} \mid 1 \leq d_\Gamma(v, w) \leq 2m + 1\}.$$

The graph Δ is again a Cayley–Abels graph for G; see Exercise 3.5. The sequence of vertices we found in our work above is a path in Δ. The orbit of v_1 under the action of H on Δ, which we denoted by O, is thus connected. We now define a graph Φ by $V\Phi := O$ and

$$E\Phi := \{\{v, w\} \mid v, w \in O \text{ and } \{v, w\} \in E\Delta\}$$

It follows that Φ is a Cayley-Abels graph for H, hence H is compactly generated by Proposition 3.2. $\square$

Let us pause to observe a technical permanence property of Cayley-Abels graphs, which we will make use of in the next section. Let G be a group acting on a set X. A **G-congruence** σ for a group G acting on a set X is an equivalence relation $\sim_\sigma$ on X such that $x \sim_\sigma y$ if and only if $g(x) \sim_\sigma g(y)$ for all $g \in G$ and $x, y \in X$. The equivalence classes of σ are called the **blocks of imprimitivity** of σ; we often call the classes "blocks" for brevity. If $x \in X$, the block containing x is denoted by x^σ.

Suppose $G \curvearrowright \Gamma$ with Γ a connected graph and σ a G-congruence on Γ. We define the quotient graph Γ/σ by setting

$$V\Gamma/\sigma := \{P \mid P \text{ is a block of } \sigma\},$$

and

$$E\Gamma/\sigma := \{\{v^\sigma, w^\sigma\} \mid \exists v', w' \in v^\sigma \ \{v, w\} \in E\Gamma \text{ and } v^\sigma \neq w^\sigma\}.$$

The action of G on Γ descends to an action on Γ/σ by $g(v^\sigma) := (g(v))^\sigma$, and this action is by graph automorphism. Additionally, Γ/σ is connected. See Exercise 3.3.

Lemma 3.16. *Let G be a t.d.l.c. Polish group and Γ be a Cayley–Abels graph for G. If σ is a G-congruence on Γ with finite blocks, then Γ/σ is a Cayley–Abels graph for G.*

Proof. The group G acts on Γ/σ vertex transitively, and since each block is finite, Γ/σ is locally finite. It remains to show the stabilizer of a vertex is compact and open.

Take $v^\sigma \in V\Gamma/\sigma$ and say that $v^\sigma = \{v_0, \dots, v_n\}$. As $G_{(v_0,\dots,v_n)} \leq G_{(v^\sigma)}$, we deduce that $G_{(v^\sigma)}$ is open. Since $G \curvearrowright \Gamma$ transitively, $v^\sigma = \{v, g_1(v), \dots, g_n(v)\}$ for some $g_1, \dots, g_n$ in G. We infer that $G_{(v^\sigma)}(v) \subseteq \{v, g_1(v), \dots, g_n(v)\}$, so

$$G_{(v^\sigma)} \subseteq G_{(v)} \cup g_1 G_{(v)} \cup \dots \cup g_n G_{(v)}.$$

The group $G_{(v^\sigma)}$ is thus also compact. $\square$

3.3 Uniqueness

The Cayley-Abels graph is unique up to a natural notion of equivalence. In the study of metric spaces, and in particular connected graphs, the notion of isometry is often too restrictive of an equivalence. There is, however, a useful weakening which captures the "large scale structure."

Definition 3.17. Let X and Y be metric spaces with metrics d_X and d_Y, respectively. A map $\phi : X \to Y$ is a **quasi-isometry** if there exist real numbers $k \geq 1$ and $c \geq 0$ for which the following hold:

1. For all $x, x' \in X$,

$$\frac{1}{k} d_X(x, x') - c \leq d_Y(\phi(x), \phi(x')) \leq k d_X(x, x') + c, \text{ and}$$

2. for all $y \in Y$, there is $x \in X$ such that $d_Y(y, \phi(x)) \leq c$

If there is a quasi-isometry between X and Y, we say they are **quasi-isometric** and write $X \simeq_{qi} Y$.

Quasi-isometries preserve the large scale structure of a metric space while allowing for bounded distortion on small scales. While not immediately obvious from the definition, the relation $\simeq_{qi}$ is an equivalence relation on metric spaces; see Exercise 3.1.

Theorem 3.18. *The Cayley–Abels graph for a compactly generated t.d.l.c. group is unique up to quasi-isometry.*

Proof. Let Γ and Δ be two Cayley–Abels graphs for a compactly generated t.d.l.c. group G. Fix $r \in V\Gamma$ and $s \in V\Delta$ and set $U := G_{(r)}$ and $V := G_{(s)}$. Lemma 3.8 supplies finite symmetric sets A_U and A_V each containing 1 such that $A_U(r) = B_1^{\Gamma}(r)$ and $A_V(s) = B_1^{\Delta}(s)$. Lemma 3.8 further ensures that $G = \langle A_U \rangle U$ and $G = \langle A_V \rangle V$.

Suppose first that $U = V$. Recalling that every vertex $u \in V\Gamma$ is of the form $g(r)$ for some $g \in G$, we define $\psi : V\Gamma \to V\Delta$ by $g(r) \mapsto g(s)$. This map is well-defined since $U = V$. For each $a \in A_U$, let $w_a v = a$ be an expression of a in the factorization $G = \langle A_V \rangle V$, where $w_a \in \langle A_V \rangle$ and $v \in V$. Take $c > \max\{|w_a| \mid a \in A_U\}$ where $|w_a|$ is the word length in the generating set A_V.

For $g, h \in G$, write $g^{-1}h = a_1 \dots a_n u$ where $a_i \in A_U$ and $u \in U = V$. We now see

$$\begin{array}{rcl} d_\Delta(\psi(g(r)), \psi(h(r))) & = & d_\Delta(g(s), h(s)) \\ & = & d_\Delta(s, g^{-1}h(s)) \\ & = & d_\Delta(s, a_1 \dots a_n(s)). \end{array}$$

We may write $a_1 \dots a_n v = w_{a_1} v_1 \dots w_{a_n} v_n$. Since $V(A_V)^n V = (A_V)^n V$ by Lemma 3.8, we may move the v_i terms past the w_{a_j} terms without changing the word length of the w_{a_j} terms; we will in general obtain a new word, however. We thus have

$$d_V(s, a_1 \dots a_n(s)) = d_V(s, w'_{a_1} \dots w'_{a_n}(s)) \leq cn = c d_\Gamma(g(r), h(r)).$$

On the other hand, for each $b \in A_V$, let $w_b u = b$ be an expression of b in the factorization $G = \langle A_U \rangle U$. Take $c' > \max\{|w_b| \mid b \in A_V\}$. As in the previous paragraph, it follows that

$$d_\Gamma(g(r), h(r)) \leq c' d_\Delta(\psi(g(r)), \psi(h(r)))$$

for any $g(r), h(r) \in V\Gamma$. Putting $k := \max\{c, c'\}$, we have

$$\frac{1}{k} d_\Gamma(g(r), h(r)) \leq d_\Delta(\psi(g(r)), \psi(h(r))) \leq k d_\Gamma(g(r), h(r)),$$

and since ψ is onto, we conclude that Γ and Δ are quasi-isometric.

We now suppose $U \leq V$. Define an equivalence relation σ on $V\Gamma$ by $g(r) \sim_\sigma h(r)$ if and only if $h^{-1}g \in V$. We see σ is indeed a G-congruence on

$V\Gamma$. Additionally, since $|V : U| < \infty$, the blocks are finite. Via Lemma 3.16, Γ/σ is a Cayley–Abels graph, and the stabilizer of r^σ is V. In view of the previous case, $\Gamma/\sigma \simeq_{qi} \Delta$. It thus suffices to show Γ/σ is quasi-isometric to Γ.

Fix $\{h_i(r) \mid i \in \mathbb{N}\}$, equivalence class representatives for $V\Gamma/\sigma$ and let c be strictly greater than the diameter of a (any) block of σ. Define $\phi : \Gamma/\sigma \to \Gamma$ by $\phi(h_i(r)^\sigma) \mapsto h_i(r)$. We outright have

$$d_{\Gamma/\sigma}\big(h_i(r)^\sigma, h_j(r)^\sigma\big) \leq d_\Gamma\big(\phi(h_i(r)^\sigma), \phi(h_j(r)^\sigma)\big).$$

On the other hand, let $v_1^\sigma, \dots, v_n^\sigma$ be a geodesic from $h_i(r)^\sigma$ to $h_j(r)^\sigma$ in Γ/σ. We may find $u_i^+ \in v_i^\sigma$ for $1 \leq i < n$ and $u_j^- \in v_j^\sigma$ for $1 < j \leq n$ such that $\{u_k^+, u_{k+1}^-\} \in E\Gamma$ for $1 \leq k < n$. We now have

$$d_\Gamma(h_i(r), h_j(r)) \leq d_\Gamma(h_i(r), u_1^+) + d_\Gamma(u_1^+, u_2^-) + d_\Gamma(u_2^-, u_2^+) + \dots + d_\Gamma(u_n^-, h_j(r)).$$

Since the blocks have diameter strictly less than c,

$$d_\Gamma(u_i^+, u_{i+1}^-) + d_\Gamma(u_{i+1}^+, u_{i+1}^-) \leq c.$$

Therefore,

$$\begin{array}{rcl} d_\Gamma(h_i(r), h_j(s)) & \leq & d_\Gamma(h_i(r), u_1^+) + c(n-2) + d_\Gamma(u_{n-1}^+, u_n^-) + d_\Gamma(u_n^-, h_j(r)) \\ & \leq & 2c + c(n-2) \leq 2cn. \end{array}$$

We conclude

$$\frac{1}{2c} d_{\Gamma/\sigma}\left(h_i(r)^\sigma, h_j(r)^\sigma\right) \leq d_\Gamma\left(\phi(h_i(r)^\sigma), \psi(h_i(r)^\sigma)\right) \leq 2c d_{\Gamma/\sigma}\left(h_i(r)^\sigma, h_j(r)^\sigma\right).$$

Since every $g(r) \in V\Gamma$ lies in some block, there is i such that $d_\Gamma(g(r), h_i(r)) \leq 2c$. Therefore, ϕ is a quasi-isometry.

The general case is now in hand: Suppose Γ and Δ are Cayley–Abels graphs for G. We may find Φ a Cayley–Abels graph for G on coset space $G/U \cap V$ by Theorem 3.5. By case two above, Γ and Δ are quasi-isometric to Φ, and therefore, they are quasi-isometric to each other. □

In view of Theorem 3.18, the quasi-isometry type of a Cayley–Abels graph of a compactly generated t.d.l.c. Polish group is an invariant of the group. We thus say t.d.l.c. Polish groups G and H are **quasi-isometric** if some (any) Cayley–Abels graph for G and H are quasi-isometric.

Proposition 3.19. *Let G be a compactly generated t.d.l.c. group. If H is a closed and cocompact subgroup of G, then H is quasi-isometric to G.*

Proof. Exercise 3.6 □

3.4 Compact presentation

In the study of finitely generated groups, finitely presented groups are of particular importance. It turns out that the notion of finite presentation generalizes to the t.d.l.c. setting.

Generalizing to the t.d.l.c. setting requires isolating a metric notion of compact presentability. For ease of discourse, we define these notions for connected graphs, but they hold in much greater generality. Let Γ be a connected graph and $c > 0$. A sequence of vertices $(v_1, \dots v_n)$ is called a c**-path** if $d_\Gamma(v_i, v_{i+1}) \leq c$ for all $1 \leq i < n$. A c-path is called a c**-loop** based at x if $v_1 = x = v_n$. Two c-loops at x are c**-elementarily homotopic** if one loop can be obtained from the other by removing one vertex. For γ a c-loop based at x, we say γ is c**-contractible** if there is a sequence of c-loops based at x $\gamma = \gamma_1, \dots, \gamma_n$ such that γ_i is c-elementarily homotopic to γ_{i+1} for all $1 \leq i < n$, and γ_n is the trivial loop (x).

Definition 3.20. For Γ a connected graph, we say that Γ is **coarsely simply connected** if for all sufficiently large c and all $x \in V\Gamma$, there is $c' \geq c$ such that every c-loop based at x is c'-contractible.

Proposition 3.21. *Let Γ and Δ be connected graphs that are quasi-isometric. Then, Γ is coarsely simply connected if and only if Δ is coarsely simply connected.*

Proof. to add □

We now isolate the notion of compact presentation.

Definition 3.22. A compactly generated t.d.l.c. Polish group is said to be **compactly presented** if some (all) Cayley–Abels graphs are coarsely simply connected.

Our definition may seem somewhat opaque, but the next theorem shows that compact presentation does indeed generalized finite presentation for discrete groups.

Theorem 3.23. *For G a compactly generated t.d.l.c. group, the following are equivalent:*

(1) G is compactly presented.

(2) There is a presentation $\langle S|R\rangle$ of G as an abstract group such that S is compact in G and there is a bound on the length of the relators in R.

Proof. to add $\square$

The usual permanence properties for finitely presented groups additionally extend to the setting of compactly presented t.d.l.c. groups. A normal subgroup N of a topological group G is compactly generated as a normal subgroup of G if there is a compact set $K \subseteq N$ such that $N = \langle gKg^{-1} \mid g \in G\rangle$.

Theorem 3.24. *Suppose $\{1\} \to K \to G \to Q \to \{1\}$ is a short exact sequence of t.d.l.c. groups.*

(1) If G is compactly presented and K is compactly generated as a normal subgroup of G, then Q is compactly presented.

(2) If G is compactly generated and Q is compactly presented, then K is compactly generated as a normal subgroup.

(3) If K and Q are compactly presented, then G is compactly presented.

Proof. to add $\square$

Notes

The notion of a Cayley–Abels graph first appeared in the work of H. Abels [1] in the early 1970s. The work of Abels, however, takes a somewhat technical approach via compactifications. There were several refinements of Abels' works in the intervening years, which eventually led to the approach given here. The approach given here should likely be attributed to B. Krön and R. Möller [11].

In several works, the Cayley–Abels graph is called the *rough Cayley graph*. The term "Cayley–Abels graphs" seems to be the accepted nomenclature.

The reader interested in a more general and deeper discussion of the geometric aspects of locally compact groups should consult [6]. The reader will find there, in particular, a general discussion of coarsely simply connected metric spaces.

3.5 Exercises

Exercise 3.1. Show the relation of quasi-isometry is an equivalence relation on the class of metric spaces.

Exercise 3.2. Prove Lemma 3.10

Exercise 3.3. Let G be a group acting on a connected graph Γ. Show that if σ is a G-congruence on Γ, then Γ/σ is connected and G acts on Γ/σ by graph automorphisms.

Exercise 3.4. Let G be a compactly generated t.d.l.c. group and Γ a be Cayley–Abels graph for G. Fix $o \in V\Gamma$ and define

$$V_n := \{g \in G \mid d_\Gamma(o, go) \leq n\}.$$

For each $n \geq 1$, show $g \in V^n$ if and only if $d_\Gamma(o, go) \leq n$. Show further V_1 is compact and a generating set for G.

Exercise 3.5. Let Γ be a vertex transitive, connected, and locally finite graph. Fix $n \geq 1$ and define the graph Γ_n by $V\Gamma_n := V\Gamma$ and

$$E\Gamma_n := \{\{v, w\} \mid 1 \leq d_\Gamma(v, w) \leq n\}.$$

(a) Show Γ_n is quasi-isometric to Γ.

(b) Show if Γ is also a Cayley–Abels graph for a t.d.l.c. group G, then Γ_n is a Cayley–Abels graph for G.

Exercise 3.6. Prove Proposition 3.19

Exercise 3.7. Suppose G is a compactly generated t.d.l.c. Polish group, $H \trianglelefteq G$ is a closed normal subgroup, and Γ is a Cayley–Abels graph for G. Show the following:

(a) The orbits of H on Γ form a G-congruence, denoted by σ.

(b) Γ/σ is locally finite and $\deg(\Gamma/\sigma) \leq \deg(\Gamma)$.

Exercise 3.8. Suppose G is compactly generated and H is a dense subset of G. Show for all $U \in \mathcal{U}(G)$, there is a finite set $F \subseteq H$ such that $G = \langle F \rangle U$. Conclude that for every dense subgroup H of G and Cayley–Abels graph Γ of G, there is a finitely generated subgroup $K \leq H$ that acts transitively on Γ.

Exercise 3.9. For G a t.d.l.c. group, define $B(G) := \{g \in G \mid \overline{g^G} \text{ is compact}\}$, where g^G is the conjugacy class of g in G.

(a) Show $B(G)$ is a characteristic subgroup of G - i.e. $B(G)$ is preserved by every topological group automorphism of G, so in particular it is normal.

(b) Suppose that G is compactly generated and fix Γ a Cayley–Abels graph for G. Show

$$B(G) = \{g \in G \mid \exists N \ \forall v \in V\Gamma \ d_\Gamma(v, g(v)) \leq N\}.$$

(c) Show that if $g \in B(G)$ is such that $\overline{\langle g \rangle}$ is compact, then $\overline{\langle g^G \rangle}$ is compact.

(d) (Challenge) Exhibit an example showing $B(G)$ need not be closed for non-compactly generated t.d.l.c. groups. (We shall see in Exercise 4.16 that $B(G)$ is closed for compactly generated G.)

Chapter 4

Essentially Chief Series

A basic concept in (finite) group theory is that of a chief factor.

Definition 4.1. A **normal factor** of a (topological) group G is a quotient K/L such that K and L are distinct (closed) normal subgroups of G with $L < K$. We say that K/L is a (topological) **chief factor** of G if there is no (closed) normal subgroup M of G such that $L < M < K$.

In finite group theory, chief factors play an essential role in the classical structure theory.

Fact 4.2. *Every finite group F admits a finite series $\{1\} = F_0 < F_1 < \dots F_n = F$ of normal subgroups of F such that each normal factor F_i/F_{i-1} is a chief factor.*

The series given in the above fact is called a chief series. Such a series additionally enjoys a uniqueness property.

Fact 4.3 (Jordan–Hölder)**.** *The chief factors appearing in a chief series of a finite group are unique up to permutation and isomorphism.*

In this chapter, we will see that compactly generated t.d.l.c. groups admit a close analogue of the chief series which additionally enjoys a uniqueness property.

4.1 Graphs revisited

4.1.1 A new definition

Our results here require a more technical, but more powerful, notion of a graph. This additional complication is necessary for the desired results to ensure the degree of a graph behaves well under quotients. The notion of a graph given here seems to be the metamathematically "correct" notion of a graph, in this author's opinion.

Definition 4.4. A **graph** $\Gamma = (V\Gamma, E\Gamma, o, r)$ consists of a set $V\Gamma$ called the **vertices**, a set $E\Gamma$ called the **edges**, a map $o : E\Gamma \to V\Gamma$ assigning to each edge an **initial vertex**, and a bijection $r : E\Gamma \to E\Gamma$, denoted by $e \mapsto \overline{e}$ and called **edge reversal**, such that $r^2 = \mathrm{id}$.

Given a classical graph, i.e. a graph as defined in Chapter 1, we produce a graph in the sense above by replacing each unordered edge $\{v, w\}$ by the ordered pairs (v, w) and (w, v). The initial vertex map o is defined to be the projection on the first coordinate, and the edge reversal map sends (v, w) to (w, v).

Convention. For the remainder of this chapter, the term "graph" shall refer to Definition 4.4.

Remark 4.5. We may see classical graphs as graphs in our sense here. Our new definition, however, allows for much more exotic graphs. For example, our new definition of a graph allows for graphs with loops and multiple edges between two vertices.

The **terminal vertex** of an edge is defined to be $t(e) := o(\overline{e})$. A **loop** is an edge e such that $o(e) = t(e)$. For e a loop, we allow both $\overline{e} = e$ and $\overline{e} \neq e$ as possibilities. For a vertex $v \in V\Gamma$, we define

$$E(v) := \{e \in E\Gamma \mid o(e) = v\} = o^{-1}(v);$$

the set $E(v)$ is sometimes called the **star** at v. The **degree** of v is $\deg(v) := |E(v)|$, and the graph Γ is **locally finite** if every vertex has finite degree. The **degree** of the graph is defined to be

$$\deg(\Gamma) := \sup_{v \in V\Gamma} \deg(v).$$

The graph is **simple** if the map $E \to V \times V$ defined by $e \mapsto (o(e), t(e))$ is injective and no edge is a loop.

A **path** p is a sequence of edges $e_1, \ldots, e_n$ such that $t(e_i) = o(e_{i+1})$ for each $i < n$. The length of the path p, denoted by $l(p)$, is the number of edges n. We say that p is a path from vertex v to vertex w if $o(e_1) = v$ and $t(e_n) = w$. A least length path between two vertices is called a **geodesic**. We say that a graph is **connected** if there is a path between any two vertices.

Connected graphs are metric spaces under the graph metric: the **graph metric** on a connected graph Γ is

$$d_\Gamma(v, u) := \begin{cases} \min\{l(p) \mid p \text{ is a path connecting } v \text{ to } u\} & \text{if } v \neq u \\ 0 & \text{if } v = u \end{cases}.$$

For $v \in V\Gamma$ and $k \geq 1$, the k**-ball** around v is defined to be $B_k(v) := \{w \in V\Gamma \mid d_\Gamma(v, w) \leq k\}$ and the k**-sphere** is defined to be $S_k(v) := \{w \in V\Gamma \mid d_\Gamma(v, w) = k\}$.

An **isomorphism** $\alpha : \Gamma \to \Delta$ between graphs is a pair (α_V, α_E) where $\alpha_V : V\Gamma \to V\Delta$ and $\alpha_E : E\Gamma \to E\Delta$ are bijections such that $\alpha_V(o(e)) = o(\alpha_E(e))$ and $\overline{\alpha_E(e)} = \alpha_E(\overline{e})$. We say that α_V and α_E respect the origin and edge reversal maps. An automorphism of Γ is an isomorphism $\Gamma \to \Gamma$. The collection of automorphisms, denoted by $\mathrm{Aut}(\Gamma)$, forms a group under the obvious definitions of composition and inversion:

$$(\alpha_V, \alpha_E) \circ (\beta_V, \beta_E) := (\alpha_V \circ \beta_V, \alpha_E \circ \beta_E) \text{ and } (\alpha_V, \alpha_E)^{-1} := (\alpha_V^{-1}, \alpha_E^{-1}).$$

The automorphism group, $\mathrm{Aut}(\Gamma)$ acts faithfully on the disjoint union $V\Gamma \sqcup E\Gamma$. As we allow for multiple edges and loops, it can be the case that the action of $\mathrm{Aut}(\Gamma)$ on $V\Gamma$ is *not faithful*. For simple graphs, the edges are completely determined by the initial and terminal vertices, so the map α_E is completely determined by α_V. In general, however, this need not be the case.

Remark 4.6. In practice, we often suppress that $g \in \mathrm{Aut}(\Gamma)$ is formally an ordered pair. This usually amounts to simply writing $g(o(e)) = o(g(e))$ and $\overline{g(e)} = g(\overline{e})$. The important bit here is that $g \in \mathrm{Aut}(\Gamma)$ acts on both $V\Gamma$ and $E\Gamma$ and these actions respect the origin and reversal maps.

Just as in Chapter 1, we make $\mathrm{Aut}(\Gamma)$ into a topological group. For finite tuples $\overline{a} := (a_1, \ldots, a_n)$ and $\overline{b} := (b_1, \ldots, b_n)$ over $V\Gamma \cup E\Gamma$, define

$$\Sigma_{\overline{a},\overline{b}} := \{g \in \mathrm{Aut}(\Gamma) \mid g(a_i) = b_i \text{ for } 1 \leq i \leq n\}.$$

The collection $\mathcal{B}$ of sets $\Sigma_{\overline{a},\overline{b}}$ as $\overline{a}$ and $\overline{b}$ run over finite sequences of elements from $V\Gamma \cup E\Gamma$ forms a basis $\mathcal{B}$ for a topology on $\mathrm{Aut}(\Gamma)$. The topology generated by $\mathcal{B}$ is called the **pointwise convergence topology**. We further recover Theorem 1.24; the proof of which is the obvious adaptation of the proof given for Theorem 1.24.

Theorem 4.7. *Let Γ be a graph. If Γ is locally finite and connected, then $\mathrm{Aut}(\Gamma)$ is a t.d.l.c. Polish group.*

4.1.2 Quotient graphs

Quotient graphs play a central role in our proof of the existence of chief series. Our more technical definition of a graph makes quotient graphs easier to define and work with; in particular, we will be able to make useful statements about the degree of quotient graphs.

Let G be a group acting on a graph Γ. For $v \in V\Gamma$ and $e \in E\Gamma$, the orbits of v and e under G are denoted by Gv and Ge, respectively. The **quotient graph** induced by the action of G, denoted by Γ/G, is defined as follows: the vertex set $V(\Gamma/G)$ is the set of G-orbits on V and the edge set $E(\Gamma/G)$ is the set of G-orbits on E. The origin map $\tilde{o} : E(\Gamma/G) \to E(\Gamma/G)$ is defined by $\tilde{o}(Ge) := Go(e)$; this is well-defined since graph automorphisms send initial vertices to initial vertices. The reversal $\tilde{r} : E(\Gamma/G) \to E(\Gamma/G)$ is given by $Ge \mapsto G\overline{e}$; this map is also well-defined. We will abuse notation and write o and r for $\tilde{o}$ and $\tilde{r}$.

There is a natural setting in which group actions descend to quotient graphs. This requires an abstract fact from permutation group theory. Recall from Chapter 1 that a G-congruence is a G-equivariant equivalence relation.

Lemma 4.8. *If G is a group acting on a set X and $N \trianglelefteq G$ is a normal subgroup, then the orbits of N on X form a G-congruence on X.*

Proof. The orbit equivalence relation on X induced by N is given by $v \sim w$ if and only if there is $n \in N$ such that $n(v) = w$. Fix $g \in G$ and suppose $v \sim w$. Letting $n \in N$ be such that $n(v) = w$, we see that $gn(v) = g(w)$, so $gng^{-1}g(v) = g(w)$. As N is normal, we conclude that $g(v) \sim g(w)$. The converse is immediate as we can act with g^{-1}. $\square$

Lemma 4.9. *Let G be a group acting on a graph Γ. If N is a normal subgroup of G, then G acts on Γ/N by $g(Nv) = Ng(v)$ and $g(Ne) = Ng(e)$.*

Furthermore, the kernel of this action of G on Γ/N contains N, so the action factors through G/N.

Proof. By Lemma 4.8, it follows that these actions are well-defined. One easily verifies that these actions respect the origin and edge reversal maps, so the action is indeed by graph automorphisms. That N acts trivially on Γ/N is immediate. □

Lemma 4.10. *Let G be a group acting on a graph Γ with N a closed normal subgroup of G and form the quotient graph Γ/N.*

(1) For $Nv \in V(\Gamma/N)$, the degree $\deg(Nv)$ equals the number of orbits of $N_{(v)}$ on $E(v)$.

(2) If $\deg(\Gamma)$ is finite, then $\deg(\Gamma/N) \leq \deg(\Gamma)$, with equality if and only if there exists a vertex $v \in V$ of maximal degree such that $N_{(v)}$ acts trivially on $E(v)$.

(3) For $v \in V$, the vertex stabilizer in G of Nv under the induced action $G \curvearrowright \Gamma/N$ is $NG_{(v)}$.

Proof. For (1), let Ne be an edge of Γ/N such that $o(Ne) = Nv$. There then exists $v' \in Nv$ and $e' \in Ne$ such that $o(e') = v'$. Letting $n \in N$ be such that $n(v') = v$, we have $o(n(e')) = v$, so $n(e') \in E(v)$. All edges of Γ/N starting at Nv are thus represented by edges of Γ starting at v. The set $E(Nv)$ thus equals $\{Ne \mid e \in E(v)\}$. Letting $\sim$ be the orbit equivalence relation of $N_{(v)}$ acting on $E(v)$, the map $\beta : E(v)/\sim \to E(Nv)$ by $[e] \mapsto Ne$ is easily verified to be a well-defined bijection. Hence, $\deg(Nv) = |E(v)/\sim|$.

For (2), claim (1) ensures $\deg(Nv) \leq \deg(v)$, and $\deg(Nv) = \deg(v)$ if and only if $N_{(v)}$ acts trivially on $E(v)$. Since $v \in V\Gamma$ is arbitrary, the conclusions for the degree of Γ/N are clear.

For (3), let H be the vertex stabilizer of Nv in Γ/N. It follows that H is simply the setwise stabilizer of Nv regarded as a subset of $V\Gamma$. In view of Lemma 4.8, the set Nv is a block of imprimitivity for the action of G on $V\Gamma$. We infer that $G_{(v)} \leq H$, so $G_{(v)} = H_{(v)}$. That N is transitive on Nv and $N \leq H$ now imply that $NG_{(v)} = H$. □

Our more technical notion of a graph ensures that Claim (1) of Lemma 4.10 holds. Let us consider an example which illustrates that this claim can fail for classical graphs and that it gives important information about the group acting on a given graph, which can be hidden in the classical setting.

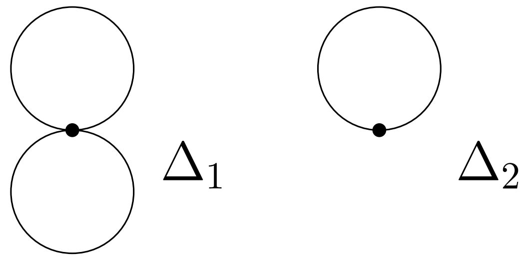

Figure 4.1: The graphs Δ_1 and Δ_2

Example 4.11. Let Γ_c be the classical graph defined by $V\Gamma_c := \mathbb{Z}$ and

$$E\Gamma_c := \{\{i, i+1\} \mid i \in \mathbb{Z}\}.$$

The group of integers $\mathbb{Z}$ and the infinite dihedral group D_∞ act on Γ_c. (The infinite dihedral group is $\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}$ where $\mathbb{Z}/2\mathbb{Z}$ acts by inversion. The generator of $\mathbb{Z}/2\mathbb{Z}$ acts on the graph Γ_c by reflection over 0.) We now compute the quotient graphs $\Gamma_c/\mathbb{Z}$ and Γ_c/D_∞. The vertex sets of both graphs consist of exactly one vertex, and since loops are not allowed, the edge sets are empty. The quotient graphs are thus the same, and we see no difference between $\mathbb{Z}$ and D_∞ from the perspective of the quotient graph.

Let us next consider the graph $\Gamma = (V\Gamma, E\Gamma, o, r)$ where $V\Gamma := \mathbb{Z}$,

$$E\Gamma := \{(i,j) \mid i,j \in \mathbb{Z} \text{ and } |i-j| = 1\},$$

the origin map is the projection onto the first coordinate, and edge reversal sends (i,j) to (j,i).

We compute the quotient graphs $\Delta_1 := \Gamma/\mathbb{Z}$ and $\Delta_2 := \Gamma/D_\infty$. The vertex sets of both graphs consist of exactly one vertex since $\mathbb{Z}$ and D_∞ both act vertex transitively. The edge set $E\Delta_1$ consists of two edges $e := \mathbb{Z}(0,1)$ and $f := \mathbb{Z}(1,0)$ such that $\overline{e} = f$. In particular, $\deg(\Delta_1) = 2$. On the other hand, $E\Delta_2$ is a singleton, since D_∞ acts edge transitively, so $\deg(\Delta_2) = 1$. See Figure 4.1. The quotient graphs, under our more technical notion of a graph, now detect a difference between $\mathbb{Z}$ and D_∞. In view of Claim (1) of Lemma 4.10, the difference detected is exactly that D_∞ has non-trivial vertex stabilizers while $\mathbb{Z}$ does not.

Lemma 4.10 shows that the degree of the quotient graph Γ/N can either become smaller or stay the same. It will be important to gain a deeper insight into the case in which the degree does not decrease under taking a quotient.

Definition 4.12. Given a group G acting on a graph Γ, we say that G acts **freely modulo kernel** on Γ if the vertex stabilizer $G_{(v)}$ acts trivially on both the vertices and the edges of Γ for all $v \in V\Gamma$.

Proposition 4.13. *Let G be a group, N be a normal subgroup of G, and Γ be a connected graph of finite degree on which G acts vertex-transitively. Then, the following are equivalent:*

(1) $\deg(\Gamma/N) = \deg(\Gamma)$.

(2) For some $v \in V\Gamma$, $N_{(v)}$ acts trivially on $E(v)$.

(3) For every $v \in V\Gamma$, $N_{(v)}$ acts trivially on $E(v)$.

(4) N acts freely modulo kernel on Γ.

Proof. (1) $\Rightarrow$ (2). Suppose that $\deg(\Gamma/N) = \deg(\Gamma)$ and fix $v \in V$. Every vertex of Γ/N and Γ has the same degree, since G acts vertex transitively on Γ. Our assumption that $\deg(\Gamma/N) = \deg(\Gamma)$ thus ensures that $|E(Nv)| = |E(v)|$. In view of Lemma 4.10, we conclude that $N_{(v)}$ acts trivially on $E(v)$.

(2) $\Rightarrow$ (3). Suppose that $N_{(v)}$ acts trivially on $E(v)$ and fix $w \in V\Gamma$. Since G acts vertex transitively, we may find $g \in G$ such that $g(v) = w$, and one verifies that $gN_{(v)}g^{-1} = N_{(w)}$, using that N is normal in G. For $e \in E(w)$ and $gng^{-1} \in gN_{(v)}g^{-1} = N_{(w)}$, we see that $g^{-1}(e) \in E(v)$, so $gng^{-1}(e) = e$. Hence, $N_{(w)}$ acts trivially on $E(w)$.

(3) $\Rightarrow$ (4). Say that $N_{(w)}$ acts trivially on $E(w)$ for every $w \in V\Gamma$. Fixing $v \in V\Gamma$, each $g \in N_{(v)}$ fixes $t(e)$, so g fixes the one sphere around v. We conclude that $N_{(v)} \leq N_{(w)}$ for each $w \in S_1(v)$. Inducting on the distance $d_\Gamma(v, w)$, we deduce that $N_{(v)} \leq N_{(w)}$ for every $w \in V\Gamma$; that Γ is connected gives us the metric d_Γ. The vertex stabilizer $N_{(v)}$ thus acts trivially on $V\Gamma$. For any $e \in E\Gamma$, the vertex stabilizer $N_{(o(e))}$ fixes e, and as $N_{(v)} \leq N_{(o(e))}$, we conclude that $N_{(v)}$ fixes e. Hence, $N_{(v)}$ acts trivially on $E\Gamma$, so N acts freely modulo kernel on Γ.

(4) $\Rightarrow$ (1). Say that N acts freely modulo kernel on Γ. Fixing $v \in V\Gamma$, the vertex stabilizer $N_{(v)}$ acts trivially on $E\Gamma$, so a fortiori, $N_{(v)}$ acts trivially on $E(v)$. Lemma 4.10 ensures that $\deg(Nv) = \deg(v)$. Since G acts vertex transitively, we deduce that $\deg(\Gamma/N) = \deg(\Gamma)$. $\square$

4.2 Chain conditions

Given a group G acting on a graph Γ and $N \trianglelefteq G$, Lemma 4.9 allows us to produce from Γ a graph on which G/N acts. For G a compactly generated t.d.l.c. group, Γ a Cayley–Abels graph for G and $N \trianglelefteq G$ closed, one hopes that Γ/N is a Cayley–Abels graph for G/N. This is indeed the case.

Proposition 4.14. *Let G be a compactly generated t.d.l.c. group with N a closed normal subgroup of G. If Γ a Cayley–Abels graph for G, then Γ/N is a Cayley–Abels graph for G/N.*

Proof. As paths in Γ induce paths in Γ/N, the graph Γ/N is connected, and G clearly acts vertex-transitively on Γ/N. Lemma 4.10 ensures that $\deg(\Gamma/N)$ is also finite, so Γ/N is connected and locally finite. Applying Lemma 4.10 a second time, we see that the vertex stabilizer of Nv in G/N is $G_{(v)}N/N$ which is compact. The graph Γ/N is therefore a Cayley–Abels graph for G/N. $\square$

A **filtering family** $\mathcal{F}$ in a partial order $(\mathcal{P}, \leq)$ is a subset of $\mathcal{P}$ such that for any $a, b \in \mathcal{F}$ there is $c \in \mathcal{F}$ for which $c \leq a$ and $c \leq b$. A **directed family** $\mathcal{D}$ is a subset of $\mathcal{P}$ such that for any $a, b \in \mathcal{D}$ there is $c \in \mathcal{D}$ for which $a \leq c$ and $b \leq c$.

We here consider filtering families and directed families of closed normal subgroups of a compactly generated t.d.l.c. group. For filtering or directed families of subgroups, the partial order is always taken to be set inclusion.

Lemma 4.15. *Let G be a compactly generated t.d.l.c. group and Γ be a Cayley–Abels graph for G.*

(1) For $\mathcal{F}$ a filtering family of closed normal subgroups of G and $M := \bigcap \mathcal{F}$, there exists $N \in \mathcal{F}$ such that $\deg(\Gamma/N) = \deg(\Gamma/M)$.

(2) For $\mathcal{D}$ a directed family of closed normal subgroups of G and $M := \overline{\langle \mathcal{D} \rangle}$, there exists $N \in \mathcal{D}$ such that $\deg(\Gamma/N) = \deg(\Gamma/M)$.

Proof. Fix $v \in V\Gamma$ and set $X := E(v)$. The action of the stabilizer $G_{(v)}$ on X induces a homomorphism $\alpha : G_{(v)} \to \mathrm{Sym}(X)$. This map is additionally continuous when $\mathrm{Sym}(X)$ is equipped with the discrete topology; see Exercise 4.4. For N a closed normal subgroup of G, the image $\alpha(N_{(v)}) =: \alpha_N$ is the subgroup of $\mathrm{Sym}(X)$ induced by $N_{(v)}$ acting on X. In view of Lemma 4.10, if $\alpha_N = \alpha_M$ for $M \trianglelefteq G$, then $\deg(\Gamma/N) = \deg(\Gamma/M)$. For a filtering or directed family $\mathcal{N} \subseteq \mathcal{N}(G)$, the family $\alpha(\mathcal{N}) := \{\alpha_N \mid N \in \mathcal{N}\}$ is a filtering or directed family of subgroups of $\mathrm{Sym}(X)$. That $\mathrm{Sym}(X)$ is a finite group ensures that $\alpha(\mathcal{N})$ is a finite family, so $\alpha(\mathcal{N})$ admits a minimum or maximum, according to whether $\mathcal{N}$ is filtering or directed.

Claim (2) is now immediate. The directed family $\alpha(\mathcal{D})$ admits a maximal element α_N. Recalling that $\alpha : G_{(v)} \to \mathrm{Sym}(X)$ is continuous, $\alpha^{-1}(\alpha_N) \cap M$ is closed, and it contains $\langle \mathcal{D} \rangle_{(v)}$ which is dense in $M_{(v)}$. Hence, $\alpha^{-1}(\alpha_N) = M_{(v)}$, and $\alpha_M = \alpha_N$. We conclude that $\deg(\Gamma/N) = \deg(\Gamma/M)$.

For claim (1), an additional compactness argument is required. If G acts freely modulo kernel on Γ, then members N and M of $\mathcal{F}$ also act freely modulo kernel. The desired result then follows since $\deg(\Gamma/N) = \deg(\Gamma) = \deg(\Gamma/M)$. Let us assume that G does not act freely modulo kernel, so $G_{(v)}$ acts non-trivially on $E(v)$ for any $v \in V\Gamma$, via Proposition 4.13.

Take $\alpha(N) \in \alpha(\mathcal{F})$ to be the minimum. Given $r \in \alpha(N)$, let Y be the set of elements of $G_{(v)}$ that *do not* induce the permutation r on X. If $r \neq 1$, then plainly $Y \neq G_{(v)}$. If $r = 1$, then $Y \neq G_{(v)}$ since $G_{(v)}$ acts non-trivially on Γ. The set Y is a proper open subset of $G_{(v)}$, and thus $G_{(v)} \setminus Y$ is a non-empty compact set.

Letting $\mathcal{K}$ be a finite subset of $\mathcal{F}$, the group $K := \bigcap_{F \in \mathcal{K}} F$ contains some element N of $\mathcal{F}$, so $\alpha(K) \geq \alpha(N)$. In particular, $K_{(v)} \not\subseteq Y$. The intersection

$$\bigcap_{F \in \mathcal{K}} (F_{(v)} \cap (G_{(v)} \setminus Y))$$

is therefore non-empty, by compactness. Hence,

$$M_{(v)} \cap (G_{(v)} \setminus Y) = \bigcap_{F \in \mathcal{F}} (F_{(v)} \cap (G_{(v)} \setminus Y)) \neq \emptyset;$$

that is, some element of $M_{(v)}$ induces the permutation r on X. Since $r \in \alpha(N)$ is arbitrary, we conclude that $\alpha(M) = \alpha(N)$, and so $\deg(\Gamma/N) = \deg(\Gamma/M)$. $\square$

In view of Proposition 4.13, the conclusion of claim (1) in Lemma 4.15 implies that the factor N/M is discrete from the point of view of the Cayley–Abels graph.

Lemma 4.16. *Let G be a compactly generated t.d.l.c. group with N a closed normal subgroup of G. If there is a Cayley–Abels graph Γ for G such that*

$\deg(\Gamma/N) = \deg(\Gamma)$, *then there exists a compact normal subgroup* L *of* G *acting trivially on* Γ *such that* L *is an open subgroup of* N.

Proof. In view of Proposition 4.13, N acts freely modulo kernel on Γ. For U the pointwise stabilizer of the star $E(v)$ for some vertex v, the subgroup U is a compact open subgroup of G, and its core K is the kernel of the action of G on Γ. Since N acts freely modulo kernel, we deduce that $N \cap U \leq K$. The group $L := K \cap N$ now satisfies the lemma. □

Combining Lemmas 4.15 and 4.16, we obtain a result that applies to compactly generated t.d.l.c. groups without dependence on a choice of Cayley–Abels graph.

Theorem 4.17. *Let* G *be a compactly generated t.d.l.c. group.*

(1) If $\mathcal{F}$ *is a filtering family of closed normal subgroups of* G*, then there exists* $N \in \mathcal{F}$ *and a closed normal subgroup* K *of* G *such that* $\bigcap \mathcal{F} \leq K \leq N$*,* $K/\bigcap \mathcal{F}$ *is compact, and* N/K *is discrete.*

(2) If $\mathcal{D}$ *is a directed family of closed normal subgroups of* G*, then there exists* $N \in \mathcal{D}$ *and a closed normal subgroup* K *of* G *such that* $N \leq K \leq \overline{\langle \mathcal{D} \rangle}$*,* K/N *is compact, and* $\overline{\langle \mathcal{D} \rangle}/K$ *is discrete.*

Proof. Fix Γ a Cayley–Abels graph for G.

For (1), suppose that $\mathcal{F}$ is a filtering family of closed normal subgroups of G and put $M := \bigcap \mathcal{F}$. Via Lemma 4.15, there is $N \in \mathcal{F}$ such that $\deg(\Gamma/N) = \deg(\Gamma/M)$. The graph Γ/M is a Cayley–Abels graph for G/M by Proposition 4.14. Furthermore, $\deg((\Gamma/M)/(N/M)) = \deg(\Gamma/N) = \deg(\Gamma/M)$; see Exercise 4.5. We may now apply Lemma 4.16 to $N/M \trianglelefteq G/M$. There is thus a closed $K \trianglelefteq G$ such that $M \leq K \leq N$, K/M is compact, and N/K is discrete.

For (2), suppose that $\mathcal{D}$ is a directed family of closed normal subgroups of G and put $L := \overline{\langle \mathcal{D} \rangle}$. Via Lemma 4.15, there is $N \in \mathcal{D}$ such that $\deg(\Gamma/L) = \deg(\Gamma/N)$. The argument now follows as in Claim (1). The quotient graph Γ/N is a Cayley-Abels graph for the quotient group G/N. Furthermore, $\deg((\Gamma/N)/(L/N)) = \deg(\Gamma/L) = \deg(\Gamma/N)$. Applying Lemma 4.16, we obtain a closed $K \trianglelefteq G$ such that $N \leq K \leq L$, K/N is compact, and L/K is discrete. □

4.3 Existence of essentially chief series

Definition 4.18. An **essentially chief series** for a topological group G is a finite series

$$\{1\} = G_0 \leq G_1 \leq \cdots \leq G_n = G$$

of closed normal subgroups such that each normal factor G_{i+1}/G_i is either compact, discrete, or a chief factor of G.

We will see that any compactly generated t.d.l.c. group admits an essentially chief series. In fact, any series of closed normal subgroups can be refined to be an essentially chief series.

Lemma 4.19. *Let G be a compactly generated t.d.l.c. group, H and L be closed normal subgroups of G with $H \leq L$, and Γ be a Cayley–Abels graph for G. Then there exists a series*

$$H =: C_0 \leq K_0 \leq D_0 \leq \cdots \leq C_n \leq K_n \leq D_n := L$$

of closed normal subgroups of G with $n \leq \deg(\Gamma/H) - \deg(\Gamma/L)$ such that

(1) for $0 \leq l \leq n$, K_l/C_l is compact, and D_l/K_l is discrete; and

(2) for $1 \leq l \leq n$, C_l/D_{l-1} is a chief factor of G.

Proof. Set $k := \deg(\Gamma/H)$ and $m := \deg(\Gamma/L)$. By recursion on i, we build a series of closed normal subgroups of G

$$H =: C_0 \leq K_0 \leq D_0 \leq \cdots \leq C_i \leq K_i \leq D_i \leq L$$

such that claims (1) and (2) hold for $0 \leq l \leq i$ and $1 \leq l \leq i$, respectively, and that there is $i \leq j \leq k - m$ for which D_i is maximal among normal subgroups of G such that $\deg(\Gamma/D_i) = k - j$ and $D_i \leq L$.

For $i = 0$, let $\mathcal{L}$ be the collection of closed normal subgroups R of G such that $H \leq R \leq L$ and $\deg(\Gamma/D_0) = k$. Via Lemma 4.15, chains in $\mathcal{L}$ admit upper bounds, so Zorn's lemma supplies D_0 a maximal element of $\mathcal{L}$. The graph Γ/H is a Cayley–Abels graph for G/H with degree k, and

$$\deg((\Gamma/H)/(D_0/H)) = \deg(\Gamma/D_0) = k.$$

Applying Lemma 4.16, we obtain a closed $K_0 \trianglelefteq G$ such that $H \leq K_0 \leq D_0$ with K_0/H compact, open, and normal in D_0/H. The groups $C_0 = H$, K_0, and D_0 satisfy the requirements of our recursive construction when $i = 0$ with $j = 0$.

Suppose we have built our sequence up to i. By construction, there is $i \leq j \leq k-m$ such that D_i is maximal with $\deg(\Gamma/D_i) = k-j$ and $D_i \leq L$. If $j = k-m$, then the maximality of D_i implies $D_i = L$, and we stop. Else, let $j' > j$ be least such that there is $M \trianglelefteq G$ with $\deg(\Gamma/M) = k - j'$ and $D_i \leq M \leq L$. Zorn's lemma in conjunction with Lemma 4.15 supply $C_{i+1} \trianglelefteq G$ minimal such that $\deg(\Gamma/C_{i+1}) = k - j'$ and $D_i < C_{i+1} \leq L$.

Consider a closed $N \trianglelefteq G$ with $D_i \leq N < C_{i+1}$. We have that

$$\deg(\Gamma/N) = \deg((\Gamma/D_i)/(N/D_i)),$$

so $\deg(\Gamma/N) \leq \deg(\Gamma/D_i) = k - j$, by Lemma 4.10. On the other hand,

$$\deg(\Gamma/C_{i+1}) = \deg((\Gamma/N)/(C_{i+1}/N)),$$

so $k - j' = \deg(\Gamma/C_{i+1}) \leq \deg(\Gamma/N)$, by Lemma 4.10. Hence, $k - j' \leq \deg(\Gamma/N) \leq k - j$.

The minimality of C_{i+1} implies that $k - j' < \deg(\Gamma/N)$. On the other hand, $j' > j$ is least such that there is $M \trianglelefteq G$ with $\deg(\Gamma/M) = k - j'$ and $D_i \leq M \leq L$, so $\deg(\Gamma/N) = k - j$. In view of the maximality of D_i, we deduce that $D_i = N$. The factor C_{i+1}/D_i is thus a chief factor of G.

Applying again Lemma 4.15, there is a closed $D_{i+1} \trianglelefteq G$ maximal such that

$$\deg(\Gamma/D_{i+1}) = k - j'$$

and $C_{i+1} \leq D_{i+1} \leq L$. Lemma 4.16 supplies a closed $K_{i+1} \trianglelefteq G$ such that $C_{i+1} \leq K_{i+1} \leq D_{i+1}$ with K_{i+1}/C_{i+1} compact and open in D_{i+1}/C_{i+1}. This completes the recursive construction.

Our recursive construction halts at some $n \leq k - m$. At this stage, $D_n = L$, verifying the theorem. □

Lemma 4.19 allows us to refine a normal series factor by factor to produce an essentially chief series. We can further bounded the length of this series in terms of a group invariant.

Definition 4.20. If G is a compactly generated locally compact group, the **degree** $\deg(G)$ of G is the smallest degree of a Cayley–Abels graph for G.

Theorem 4.21 (Reid–W.). *Suppose that G is a compactly generated t.d.l.c. group. If $(G_i)_{i=1}^{m-1}$ is a finite ascending sequence of closed normal subgroups of G, then there exists an essentially chief series for G*

$$\{1\} = K_0 \leq K_1 \leq \cdots \leq K_l = G,$$

such that $\{G_1, \ldots, G_{m-1}\}$ is a subset of $\{K_0, \ldots, K_l\}$ and $l \leq 2m + 3\deg(G)$. Additionally, at most $\deg(G)$ of the factors K_{i+1}/K_i are neither compact nor discrete.

Proof. Let us extend the series $(G_i)_{i=1}^{m-1}$ by $G_0 := \{1\}$ and $G_m := G$ to obtain the series

$$\{1\} =: G_0 \leq G_1 \leq \cdots \leq G_{m-1} \leq G_m := G.$$

Fix Γ a Cayley-Abels graph for G such that $\deg(G) = \deg(\Gamma)$. For each $j \in \{0, \ldots, m-1\}$, we apply Lemma 4.19 to $L := G_{j+1}$ and $H := G_j$. This produces the essentially chief series $\{1\} = K_0 \leq K_1 \leq \cdots \leq K_l = G$ for G. We now argue that l has the claimed bound.

For each $0 \leq j \leq m$, put $k_j := \deg(\Gamma/G_j)$. In view of Lemma 4.19, the number of new normal subgroups added strictly between G_j and G_{j+1} is at most $3(k_j - k_{j+1}) + 1$, and at most $k_j - k_{j+1}$ of the factors are neither compact nor discrete. The total number of terms in the essentially chief series not including G_m is thus at most

$$\begin{aligned} \textstyle\sum_{j=0}^{m-1}(3(k_j - k_{j+1}) + 2) &= 2m + 3(\deg(\Gamma) - \deg(\Gamma/G)) \\ &\leq 2m + 3\deg(G), \end{aligned}$$

and the total number of non-compact, non-discrete factors is at most

$$\sum_{j=0}^{m-1}(k_j - k_{j+1}) \leq \deg(G).$$

It now follows that $l \leq 2m + 3\deg(G)$.

□

Corollary 4.22 (Existence of essentially chief series). *Every compactly generated t.d.l.c. group admits an essentially chief series.*

4.4 Uniqueness of essentially chief series

The uniqueness result for essential chief series takes much more work than the existence theorem, and it is, although not obviously, one of the deepest results so far. The uniqueness property will allow us in Chapter **??** to make striking general statements about normal subgroups of t.d.l.c. Polish groups and in particular uncover the structure of chief factors.

Isomorphism is too restrictive of an equivalence for chief factors in Polish groups. The problem with isomorphism arises from a subtlety in the second isomorphism theorem. For G a Polish group and K and L closed normal subgroups of G, the second isomorphism theorem states that $KL/L \simeq K/K\cap L$ as abstract groups. This statement *does not* hold in a topological sense in the setting of Polish or locally compact groups. The internal product KL is not in general closed, so KL/L fails to be a Polish or locally compact group. We develop a weaker notion of equivalence called association. The relation of association "fixes" the second isomorphism theorem for Polish or locally compact groups by relating $K/K\cap L$ to $\overline{KL}/L$, instead of relating $K/K\cap L$ to KL/L.

For G a group and K/L a normal factor of G, the **centralizer** of K/L in G is defined to be

$$\mathrm{C}_G(K/L) := \{g \in G \mid \forall k \in K\ [g,k] \in L\}$$

where $[g,k]$ is the commutator $gkg^{-1}k^{-1}$. Given a subgroup H of G, we put $\mathrm{C}_H(K/L) := \mathrm{C}_G(K/L) \cap H$.

Definition 4.23. For a topological group G, closed normal factors K_1/L_1 and K_2/L_2 are **associated** if $\mathrm{C}_G(K_1/L_1) = \mathrm{C}_G(K_2/L_2)$.

The association relation is clearly an equivalence relation on normal factors. There is furthermore a key refinement theorem, from which we deduce our uniqueness result. The proof of this theorem is rather technical and requires several new notions, so we delay the proof until the next section.

Theorem 4.24. *Let G be a t.d.l.c. Polish group and K/L be a non-abelian chief factor of G. If*

$$\{1\} = G_0 \leq G_1 \leq \cdots \leq G_n = G$$

is a series of closed normal subgroups in G, then there is exactly one $i \in \{0,\dots,n-1\}$ for which there exist closed normal subgroups $G_i \leq B \leq A \leq$

G_{i+1} of G such that A/B is a non-abelian chief factor associated to K/L. Specifically, this occurs for the least $i \in \{0, \dots, n-1\}$ such that $G_{i+1} \not\leq \mathrm{C}_G(K/L)$.

Definition 4.25. For G a Polish group and K/L a chief factor of G, we say that K/L is **negligible** if K/L is either abelian or associated to a compact or discrete chief factor.

Negligible chief factors look compact or discrete from the point of view of association, and in our uniqueness theorem, we must ignore these factors. We shall see later that negligible chief factors are either close to compact or close to discrete.

In contrast to the results about existence of chief series, we need not assume that G is compactly generated for our uniqueness theorem, but we do need to assume the group is Polish.

Theorem 4.26 (Reid–W.)**.** *Suppose that G is an t.d.l.c. Polish group and that G has two essentially chief series $(A_i)_{i=0}^m$ and $(B_j)_{j=0}^n$. Define*

$$I := \{i \in \{1, \dots, m\} \mid A_i/A_{i-1} \text{ is a non-negligible chief factor of } G\}; \text{and}$$
$$J := \{j \in \{1, \dots, n\} \mid B_j/B_{j-1} \text{ is a non-negligible chief factor of } G\}.$$

Then there is a bijection $f : I \to J$ where $f(i)$ is the unique element $j \in J$ such that A_i/A_{i-1} is associated to B_j/B_{j-1}.

Proof. Theorem 4.24 provides a function $f : I \to \{1, \dots, n\}$ where $f(i)$ is the unique element of $\{1, \dots, n\}$ such that A_i/A_{i-1} is associated to a non-abelian chief factor C/D such that $B_{f(i)} \leq D \leq C \leq B_{f(i)-1}$.

If $B_{f(i)}/B_{f(i)-1}$ is compact, discrete, or abelian, $C/B_{f(i)}$ is compact, discrete or abelian, as each of these classes of groups is stable under taking closed subgroups. Since these classes are also stable under quotients, C/D is either compact, discrete, or abelian. The chief factor C/D is non-abelian, so it must be the case that $B_{f(i)}/B_{f(i)-1}$ is non-abelian. On the other hand, A_i/A_{i-1} is associated to C/D, hence, C/D is neither compact nor discrete, since A_i/A_{i-1} is non negligible. We thus deduce that $B_{f(i)}/B_{f(i)-1}$ is chief, so A_i/A_{i-1} is associated to $B_{f(i)}/B_{f(i)-1}$. Since association is an equivalence relation, we conclude that $B_{f(i)}/B_{f(i)-1}$ is non-negligible, and therefore, $f(i) \in J$.

We thus have a well-defined function $f : I \to J$. The same argument with the roles of the series reversed produces a function $f' : J \to I$ such that

B_j/B_{j-1} is associated to $A_{f'(j)}/A_{f'(j)-1}$. Since each factor of the first series is associated to at most one factor of the second by Theorem 4.24, we conclude that f' is the inverse of f, hence f is a bijection. □

Corollary 4.27 (Uniqueness of essentially chief series). *The non-negligible chief factors appearing in an essentially chief series of a compactly generated t.d.l.c. Polish group are unique up to permutation and association.*

4.5 The refinement theorem

4.5.1 Normal compressions

Definition 4.28. Let G and H be topological groups. A continuous homomorphism $\psi : G \to H$ is a **normal compression** if it is injective with a dense and normal image. When the choice of ψ is unimportant, we say that H is a normal compression of G.

Normal compressions arise naturally in the study of normal subgroups of topological groups. Say that G is a topological group with K and L closed normal subgroups of G. The map $\psi : K/K \cap L \to \overline{KL}/L$ by $k(K \cap L) \mapsto kL$ is a continuous homomorphism with image KL/L. Hence, ψ is a normal compression, and it is not onto as soon as KL is not closed in G.

Lemma 4.29. *Let G and H be t.d.l.c. Polish groups with $\psi : G \to H$ a normal compression. For any $h \in H$, the map $\phi_h : G \to G$ defined by*

$$\phi_h(g) := \psi^{-1}(h\psi(g)h^{-1})$$

is a topological group automorphism of G.

Proof. We leave it to the reader to verify that ϕ_h is an automorphism of G as an abstract group; see Exercise 4.9. To show that ϕ_h is a topological group automorphism, it suffices to argue that ϕ_h is continuous at 1.

Fixing $U \subseteq G$ a compact open subgroup, we see

$$\begin{aligned} \phi_h^{-1}(U) &= \{g \in G \mid \psi^{-1}(h\psi(g)h^{-1}) \in U\} \\ &= \psi^{-1}(h^{-1}\psi(U)h). \end{aligned}$$

Since ψ is continuous, $\psi(U)$ is compact and so closed. Thus, $W := \psi^{-1}(h^{-1}\psi(U)h)$ is a closed set.

The set W is indeed a closed subgroup of G. Furthermore, U has countable index in G, so W also has countable index in G. Write $G = \bigcup_{i\in\mathbb{N}} g_iW$. The Baire category theorem, Fact 1.10, implies that g_iW is non-meagre for some i. As multiplication by g_i is a homeomorphism of G, we infer that W is non-meagre. The subgroup W is thus somewhere dense, so W has non-empty interior, as it is closed. We conclude that W is open and that ϕ_h is continuous. $\square$

In view of Lemma 4.29, there is a canonical action of H on G.

Definition 4.30. Suppose that G and H are t.d.l.c. Polish groups and $\psi : G \to H$ is a normal compression. We call the action of H on G given by $h.g := \phi_h(g)$ the **ψ-equivariant action** of G on H. When clear from context, we suppress "ψ."

The name ψ-equivariant action is motivated by the following lemma.

Lemma 4.31. *Suppose that G and H are t.d.l.c. Polish groups and $\psi : G \to H$ is a normal compression. Letting H act on G by the ψ-equivariant action and H act on itself by conjugation, the map $\psi : G \to H$ is H-equivariant. That is to say, $\psi(h.g) = h\psi(g)h^{-1}$ for all $h \in H$ and $g \in G$.*

Proof. Exercise 4.12 $\square$

We now argue that the ψ-equivariant action is continuous. This requires a technical lemma.

Lemma 4.32. *Let G and H be t.d.l.c. Polish groups with $\psi : G \to H$ a normal compression. For $U \in \mathcal{U}(G)$ and $g \in G$, $N_H(\psi(gU))$ is open in H.*

Proof. We first argue that $N_H(\psi(U))$ is open. The group G is second countable, so G has countably many compact open subgroups. Lemma 4.29 ensures that $h.U$, where the action is the ψ-equivariant action, is also a compact open subgroup for any $h \in H$. It now follows that $\mathrm{Stab_H(U)}$ has countable index in H.

Take $h \in \mathrm{Stab_H(U)}$. For $u \in \psi(U)$, we see that $\psi^{-1}(h\psi(u)h^{-1}) \in U$. Hence, $h\psi(u)h^{-1} \in \psi(U)$, so $\mathrm{Stab_H(U)} \leq \mathrm{N_H(\psi(U))}$. The group $N_H(\psi(U))$ thus has countable index in H, and via the Baire category theorem, it follows that $N_H(\psi(U))$ is open in H.

Put $L := N_H(\psi(U))$. Given a coset kU and $l \in U$,

$$l.(kU) = \psi^{-1}(l\psi(k)\psi(U)l^{-1}) = \psi^{-1}(l\psi(k)l^{-1}\psi(U)) = k'U$$

for some $k' \in G$. We thus obtain an action of L on $\{kU \mid k \in G\}$. There are only countably many left cosets kU of U in G, so $\mathrm{Stab_L(gU)}$ has countable index in L.

As in the second paragraph, $N_L(\psi(gU))$ has countable index in L and is closed. Hence, $N_L(\psi(gU))$ is open in L, and so, $N_H(\psi(gU))$ is open in H. $\square$

Proposition 4.33. *If G and H are t.d.l.c. Polish groups and $\psi : G \to H$ is a normal compression, then the ψ-equivariant action is continuous.*

Proof. Let $\alpha : H \times G \to G$ by $(h, g) \mapsto h.g$ be the action map. The topology on G has a basis consisting of cosets of compact open subgroups. It thus suffices to show $\alpha^{-1}(kU)$ is open for any $k \in G$ and $U \in \mathcal{U}(G)$.

Fix $k \in G$ and $U \in \mathcal{U}(G)$ and let $(h, g) \in \alpha^{-1}(kU)$. Lemma 4.29 ensures the map ψ_h is continuous. There is then $W \in \mathcal{U}(G)$ such that $h.(gW) = \phi_h(gW) \subseteq kU$. Additionally, Lemma 4.32 tells us that $L := N_H(\psi(gW))$ is open.

We now consider the open neighborhood $hL \times gW$ of (h, g). For $(hl, gw) \in hL \times gW$,

$$\begin{aligned} \alpha(hl, gw) = hl.gw &= \psi^{-1}(hl\psi(g)\psi(w)l^{-1}h^{-1}) \\ &= \psi^{-1}(h\psi(g)\psi(w')h^{-1}) \\ &= h.(gw'). \end{aligned}$$

The element $h.(gw')$ in kU, so $\alpha(hl, gw) \in kU$. We conclude that $\alpha(hL \times gW) \subseteq kU$, and thus α is continuous. $\square$

In view of Proposition 1.27, Proposition 4.33 allows us to conclude the semidirect product $G \rtimes H$ is a t.d.l.c. Polish group. To emphasize the ψ-equivariant action, we denote this semidirect product by $G \rtimes_\psi H$. If $O \leq H$ is a subgroup, we can form the semi-direct product $G \rtimes_\psi O$ by restricting the action of H to O.

Our next theorem gives a natural factorization of a normal compression.

Theorem 4.34. *Suppose that G and H are t.d.l.c. Polish groups and $\psi : G \to H$ is a normal compression. For $U \leq H$ an open subgroup, the following hold:*

(1) $\pi : G \rtimes_\psi U \to H$ via $(g, u) \mapsto \psi(g)u$ is a continuous surjective homomorphism with $\ker(\pi) = \{(g^{-1}, \psi(g)) \mid g \in \psi^{-1}(U)\}$*;*

(2) $\psi = \pi \circ \iota$ *where* $\iota : G \to G \rtimes_\psi U$ *is the usual inclusion; and*

(3) $G \rtimes_\psi U = \overline{\iota(G)\ker(\pi)}$*, and the subgroups* $\iota(G)$ *and* $\ker(\pi)$ *are closed normal subgroups of* $G \rtimes_\psi U$ *with trivial intersection.*

Proof. (1) The image of π is $\psi(G)U$. As $\psi(G)$ is dense and U is an open subgroup, it follows that $\psi(G)U = H$. Hence, π is surjective. By definition,

$$(g, u)(g', u') = (g \cdot u.g', uu').$$

In view of Lemma 4.31, we see that

$$\begin{aligned}\pi(g \cdot u.g', uu') = \psi(g \cdot (u.g'))uu' &= \psi(g)u\psi(g')u^{-1}uu' \\ &= \psi(g)u\psi(g')u' \\ &= \pi(g, u)\pi(g', u').\end{aligned}$$

Hence, π is a homomorphism. To see that π is continuous, it suffices to check that π is continuous at 1. Take $V \leq U$ a compact open subgroup of H. The preimage $\pi^{-1}(V)$ contains $\psi^{-1}(V) \times V$ which is an open neighborhood of 1. Hence, π is continuous at 1. Finally, an easy calculation shows $\ker(\pi) = \{(g^{-1}, \psi(g)) \mid g \in \psi^{-1}(U)\}$. Claim (1) is thus demonstrated.

Claim (2) is immediate.

(3) By Claim (1), $\iota(G) = \{(g, 1) \mid g \in G\}$ intersects $\ker(\pi)$ trivially, and both $\iota(G)$ and $\ker(\pi)$ are closed normal subgroups. The product $\iota(G)\ker(\pi)$ is dense, since it is a subgroup containing the set $\{(1, h) \mid h \in \psi(G) \cap U\} \cup \iota(G)$. We have thus verified (3). □

The factorization established in Theorem 4.34 allows us to make statements about the relationship between normal subgroups of G and H, when there is a normal compression $\psi : G \to H$. These results will be essential in establishing the key refinement theorem.

Proposition 4.35. *Let* G *and* H *be t.d.l.c. Polish groups,* $\psi : G \to H$ *be a normal compression, and* K *be a closed normal subgroup of* G*.*

(1) The image $\psi(K)$ *is a normal subgroup of* H*.*

(2) If $\psi(K)$ *is also dense in* H*, then* $\overline{[G, G]} \leq K$*, and every closed normal subgroup of* K *is normal in* G*.*

Proof. Form the semidirect product $G \rtimes_\psi H$, let $\iota : G \to G \rtimes_\psi H$ be the usual inclusion, and let $\pi : G \rtimes_\psi H \to H$ be the map given in Theorem 4.34.

(1) The intersection $\ker(\pi) \cap \iota(G)$ is trivial, so $\ker(\pi)$ centralizes $\iota(G)$, since each of the groups is normal in $G \rtimes_\psi H$. In particular, $\ker(\pi)$ normalizes $\iota(K)$. The normalizer $N_{G \rtimes_\psi H}(K)$ therefore contains the dense subgroup $\iota(G)\ker(\pi)$. As $\iota(K)$ is a closed subgroup of $G \rtimes_\psi H$, we conclude that $\iota(K) \trianglelefteq G \rtimes_\psi H$, since normalizers of closed subgroups are closed. Theorem 4.34 now ensures that $\pi(\iota(K)) = \psi(K)$ is normal in H.

(2) Since π is a quotient map and $\psi(K) = \pi(\iota(K))$ is dense in H, it follows that $\iota(K)\ker(\pi)$ is dense in $G \rtimes_\psi H$. Claim (1) implies that $\iota(K)$ is a closed normal subgroup of $G \rtimes_\psi H$. The image of $\ker(\pi)$ is thus dense under the usual projection $\chi : G \rtimes_\psi H \to G \rtimes_\psi H/\iota(K)$. On the other hand, Theorem 4.34 ensures $\iota(G)$ and $\ker(\pi)$ commute, hence $\iota(G)/\iota(K)$ has dense centralizer in $G \rtimes_\psi H/\iota(K)$. The group $\iota(G)/\iota(K)$ is then central in $G \rtimes_\psi H/\iota(K)$, so in particular, $\iota(G)/\iota(K)$ is abelian. We conclude that G/K is abelian and $\overline{[G,G]} \leq K$.

Let M be a closed normal subgroup of K. The map $\psi \restriction_K : K \to M$ is a normal compression map. Applying part (1) to the compression map $\psi \restriction_K$, we see that $\psi(M)$ is normal in H, so in particular $\psi(M)$ is normal in $\psi(G)$. Since ψ is injective, M is in fact normal in G. $\square$

For G a topological group and A a group acting on G by automorphisms, say that G is A-**simple** if A leaves no proper non-trivial closed normal subgroup of G invariant. For example, G is $\{1\}$-simple if and only if G is topologically simple.

Theorem 4.36. *Suppose that G and H are non-abelian t.d.l.c. Polish groups with $\psi : G \to H$ a normal compression and suppose that G and H admit actions by topological group automorphisms of a (possibly trivial) group A such that ψ is A-equivariant.*

(1) If G is A-simple, then so is $H/Z(H)$, and $Z(H)$ is the unique largest proper closed A-invariant normal subgroup of H.

(2) If H is A-simple, then so is $\overline{[G,G]}$, and $\overline{[G,G]}$ is the unique smallest non-trivial closed A-invariant normal subgroup of G.

Proof. (1) Let L be a proper closed normal A-invariant subgroup of H. Clearly, $\psi(G) \not\leq L$, so $\psi^{-1}(L)$ is a proper closed normal A-invariant subgroup of G and hence is trivial. The subgroups $\psi(G)$ and L are then normal subgroups of H with trivial intersection, so $\psi(G)$ and L commute. Since $\psi(G)$ is dense in H and centralizers are closed, $L \leq Z(H)$. In particular, $H/Z(H)$ does not have any proper non-trivial closed normal A-invariant subgroup, and (1) follows.

(2) The subgroup $L := \overline{[G,G]}$ is preserved by every topological group automorphism of G and hence is normal and A-invariant; note $L \neq \{1\}$, since G is not abelian. The image $\psi(L)$ is therefore a non-trivial A-invariant subgroup of H and hence is dense. Proposition 4.35 further implies any closed A-invariant normal subgroup of L is normal in G.

Letting K be an arbitrary non-trivial closed A-invariant normal subgroup of G, Proposition 4.35 ensures the group $\psi(K)$ is normal in H. Since ψ is A-equivariant, $\psi(K)$ is indeed an A-invariant subgroup of H, so by the hypotheses on H, the subgroup $\psi(K)$ is dense in H. Applying Proposition 4.35 again, we conclude that $K \geq L = \overline{[G,G]}$. The subgroup L is thus the unique smallest non-trivial closed A-invariant normal subgroup of G, and (2) follows. □

4.5.2 The proof

Lemma 4.37. *For K and L closed normal subgroups of a topological group G, the map $\phi : K/(K \cap L) \to \overline{KL}/L$ via $k(K \cap L) \mapsto kL$ is a G-equivariant normal compression map, where G acts on each group by conjugation.*

Proof. Exercise 4.13 □

Lemma 4.38. *Let K_1/L_1 and K_2/L_2 be closed normal factors of a topological group G and let G act on each factor by conjugation. If $\psi : K_1/L_1 \to K_2/L_2$ is a G-equivariant normal compression, then $\mathrm{C}_G(K_2/L_2) = \mathrm{C}_G(K_1/L_1)$.*

Proof. Exercise 4.14. □

Proof of Theorem 4.24. We leave the uniqueness of i to the reader in Exercise 4.15.

For existence, let $\alpha : G \to \mathrm{Aut}(K/L)$ be the homomorphism induced by the conjugation action of G on K/L. Since K/L is centerless, the normal subgroup $\mathrm{Inn}(K/L)$ is isomorphic as an abstract group to K/L. Every non-trivial subgroup of $\alpha(G)$ normalized by $\mathrm{Inn}(K/L)$ also has non-trivial intersection with $\mathrm{Inn}(K/L)$, since the centralizer of $\mathrm{Inn}(K/L)$ in $\alpha(G)$ is trivial.

Take i minimal such that $G_{i+1} \not\leq \mathrm{C}_G(K/L)$. The group $\alpha(G_{i+1})$ is then non-trivial and normalized by $\mathrm{Inn}(K/L)$, so $\mathrm{Inn}(K/L) \cap \alpha(G_{i+1})$ is non-trivial. Set

$$B := \mathrm{C}_{G_{i+1}}(K/L), \; R := \alpha^{-1}(\mathrm{Inn}(K/L)) \cap G_{i+1}, \text{ and } A := \overline{[R,K]B}.$$

The groups A and B are closed normal subgroups of G such that $G_i \leq B \leq A \leq G_{i+1}$.

Since $\mathrm{Inn}(K/L) \cap \alpha(G_{i+1})$ is non-trivial, there are non-trivial inner automorphisms of K/L induced by the action of R, so $\overline{[R,K]} \not\leq L$. Since K/L is a chief factor of G, it must be the case that $K = \overline{[R,K]L}$. If A/B is abelian, then $[A,A] \leq C_G(K/L)$, so $[[R,K],[R,K]]$ centralizes K/L. As K/L is topologically perfect, it follows that K/L has a dense center, so K/L is abelian, which is absurd. The closed normal factor A/B is thus non-abelian.

Set $C := \mathrm{C}_G(K/L)$ and $M := \overline{KC}$. We see that $K \cap C = L$ since K/L is centerless and that $A \cap C = B$ from the definition of B. As $K = \overline{[R,K]L}$, it is also the case that $\overline{AL} = \overline{KB}$, and thus,

$$M = \overline{KC} = \overline{KBC} = \overline{ALC} = \overline{AC}.$$

We are now in position to apply Lemma 4.37 and thereby obtain G-equivariant normal compression maps $\psi_1 : K/L \to M/C$ and $\psi_2 : A/B \to M/C$.

Lemma 4.31 implies that $\mathrm{C}_G(M/C) = \mathrm{C}_G(K/L) = C$, so M/C is centerless. The factor K/L is chief, and thus, it has no proper G-invariant closed normal subgroups. Theorem 4.36 ensures that M/C also has no G-invariant closed normal subgroups; that is to say, M/C is a chief factor of G. Applying Theorem 4.36 to ψ_2, the group $D := \overline{[A,A]B}$ is such that D/B is the unique smallest non-trivial closed G-invariant subgroup of A/B. In particular, D/B is a chief factor of G.

The map ψ_2 restricts to a G-equivariant compression from D/B to M/C, so $\mathrm{C}_G(D/B) = \mathrm{C}_G(M/C)$. Since M/C is non-abelian, D/B is also non-abelian. We conclude that $\mathrm{C}_G(D/B) = \mathrm{C}_G(K/L)$, and hence D/B is a non-abelian chief factor of G associated to K/L with $G_{i+1} \leq B < D \leq G_i$. The proof is now complete. □

Notes

The first hints of the essentially chief series seem to appear in the work of V.I. Trofimov [14]. Moreover, in loc. cit., Trofimov makes the crucial observation that quotienting a Cayley–Abels graph by a normal subgroup can only drop the degree. Independently and rather later, Burger–Mozes analyze the normal subgroups of certain t.d.l.c. groups acting on trees in [4] and in particular find minimal non-trivial closed normal subgroups. In [5], Caprace–Monod push parts of the analysis of Burger–Mozes much further. Finally, Reid and the author complete the story in [13].

Theorem 4.24 in fact holds for all Polish groups. We restrict our attention to the case of t.d.l.c. Polish groups to avoid appealing to facts from descriptive set theory. The interested reader can find the general statement and proof of Theorem 4.24 in [12].

4.6 Exercises

Exercise 4.1. Verify that the origin map and the edge reversal maps are well-defined in a quotient graph.

Exercise 4.2. Give a complete proof of Lemma 4.9.

Exercise 4.3. Show that the actions defined in Lemma 4.9 respect the origin and edge reversal maps.

Exercise 4.4. Let Γ be a locally finite graph and $G \leq \mathrm{Aut}(\Gamma)$ a closed subgroup. Fix $v \in V\Gamma$ and set $X := E(v)$. Show the homomorphism $\alpha : G_{(v)} \to \mathrm{Sym}(X)$ induced by the action of $G_{(v)}$ on X is continuous when $\mathrm{Sym}(X)$ is equipped with the discrete topology.

Exercise 4.5. Let G be a group acting on a graph Γ and suppose that $M \leq N$ are normal subgroups of G. Show there is a G-equivariant graph isomorphism between $(\Gamma/M)/(M/N)$ and Γ/N.

Exercise 4.6. Suppose that K/L is closed normal factor of a topological group G. Show $\overline{[K/L, K/L]} = \overline{[K,K]L}/L$.

Exercise 4.7. Let G be a topological group and K/L be a closed normal factor of G. Suppose that $D \subseteq G$ is such that DL/L is a dense subset of K/L. Show that if $g \in G$ is such that $[g, d] \in L$ for all $d \in D$, then $g \in \mathrm{C}_G(K/L)$.

Exercise 4.8. Let G be a group with K/L a closed normal factor of G. Taking $\pi : G \to G/L$ to be the usual projection, show $\mathrm{C}_G(K/L) = \pi^{-1}(\mathrm{C}_{G/L}(K/L))$.

Exercise 4.9. Let G and H be groups with $\psi : G \to H$ an injective homomorphism such that $\psi(G)$ is normal in H. Show the map $\phi_h : G \to G$ defined by $\phi_h(g) := \psi^{-1}(h\psi(g)h^{-1})$ is group automorphism of G for any $h \in H$.

Exercise 4.10. Let $\psi : G \to H$ be a normal compression of topological groups. Verify that the ψ-equivariant action is a group action of G on H.

Exercise 4.11. Let G and H be t.d.l.c. Polish groups with $\psi : G \to H$ a normal compression. Show every closed normal subgroup of G is invariant under the ψ-equivariant action of H on G.

Exercise 4.12. Prove Lemma 4.31.

Exercise 4.13. Prove Lemma 4.37.

Exercise 4.14. Prove Lemma 4.38.

Exercise 4.15. Verify the uniqueness claim of Theorem 4.24.

Exercise 4.16 (Trofimov; Möller)**.** For G a t.d.l.c. group, recall from Exercise 3.9 that $B(G) = \{g \in G \mid \overline{g^G} \text{ is compact}\}$, where g^G is the conjugacy class of g in G. Show $B(G)$ is closed for G a compactly generated t.d.l.c. group. **HINT:** Use Exercise 3.9 and Theorem 4.17.

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